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ABSTRACT

This guide outlines the curriculum for a ninth-year mathematics course for students not prepared to cope with the usual first-year algebra course. It is intended to provide personal relevance for these students by including supplementary units on probability and statistics, slide rule use, flow charting and use of calculators, consumer mathematics, informal geometry and mathematical reasoning. The major goal of the course is to develop mathematics skills and competencies which will ensure student success in algebra. The unit topics for this purpose are: graphing; number bases; set of integers; rational numbers; metric geometry; and ratio, proportion, and percent. (JP)

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INTRODUCTION TO HIGH SCHOOL Mathematics

GRADE 9-COURSE 2

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INTRODUCTION TO HIGH SCHOOL

MATHEMATICS

GRADE 9 — COURSE 2

The University of the State of New York/The State Education Department
Bureau of Secondary Curriculum Development/Albany/1970

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FOREWORD

Many students who enter the ninth grade are not ready to cope with the recommended State course outlined in the syllabus *Sixth Year Mathematics, Course 1 - Algebra*. Rather than schedule them into that course, it is generally more desirable to provide a prealgebra course. The revised course of study, *Introduction to High School Mathematics*, is intended to provide students with a format that has personal relevance and a goal of developing mathematics skills and competencies which will encourage students to explore further the field of mathematics on a personal or formal basis.

A prototype edition of this course of study was prepared in August 1969, and copies were distributed to 30 selected junior and senior high schools for an inschool trial during the school year 1969-70.

The members of the Mathematics 9 Curriculum Committee, who provided the guidelines for the prototype edition were: Joseph Gehringer, Charles Sumner Junior High School, New York City; Elmer Heinecke, Wantagh High School; Thomas Huestis, Niagara Falls Public Schools; Gloria Kelly, James Farley Junior High School, Stony Point; and William Vanderhoof, Gates-Chili Senior High School, Rochester.

The original content and approach materials were written by Frank Cunningham, Hicksville Junior High School and Larry Lewis, Gates-Chili Central School, Rochester, who brought the recommendations of the committee into focus in the prototype form.

During the spring of 1970, feedback information concerning the prototype course of study was acquired through the use of questionnaires which were sent to participating schools and teachers. In addition, a second ad hoc committee was impaneled to consider a revision. Since this committee included teachers who had participated in the tryout of this course, many worthwhile recommendations were made.

The members of the second Mathematics 9 Curriculum Committee were: Evelyn Brenzel, Hurlbut W. Smith Junior High School, Syracuse; Elizabeth Bukanz, Walt Whitman Junior High School, Yonkers; Joseph Gehringer, Charles Sumner Junior High

School, New York City; Farroh Hormozi, Hicksville Junior High School; Richard Story, Great Neck North Junior High School; and Doris Steminsky, Burnt Hills-Ballston Lake Junior High School, Burnt Hills.

The second set of writers were Carl Elsholz, A. J. Veraldi Junior High School, Middletown, and Marlene Steger, Walt Whitman Junior High School, Yonkers, who incorporated the recommendations.

This document was prepared under the direction of Bruno B. Baker, Associate in the Bureau of Mathematics Education. Important suggestions were made by Aaron L. Buchman and Fredric Paul, Associates in the Bureau of Mathematics Education. Frank S. Hawthorne, Chief of the Bureau of Mathematics Education, had overall responsibility for maintaining articulation with other segments of the mathematics program. Curriculum responsibility for the project was assigned to Robert F. Zimmerman, Associate in Secondary Curriculum, who worked with the committee and mathematics staff in planning the publication. The final draft was prepared and edited by Mr. Baker, Mr. Buchman, and the undersigned.

Gordon E. Van Hooft
Chief, Bureau of Secondary Curriculum Development

William E. Young
Director, Curriculum Development Center

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INTRODUCTION TO HIGH SCHOOL MATHEMATICS

GRADE 9

1. Introductory Information for the Teacher

In the ninth grade, it is necessary to present an alternate course in mathematics for those students whose needs, interests, and abilities can best be met by a program other than algebra. This type of course may be considered either as a prealgebra course or as a terminal course in high school mathematics.

- A key purpose in the development of this ninth grade course is to present to the students a challenging and practical mathematical experience. An attempt has been made in this development to present material, ideas, and situations which will help to stimulate student interest and help to overcome many of the arithmetic deficiencies a student may have. Further growth in the usage and understanding of mathematical skills and concepts can be enhanced through practice and a variety of associated activities which are of a significant nature to the student.

- It is suggested that, whenever possible, the usage of extensive rigor and formalized definitions be avoided. For many students, formality of structure tends to be completely uninteresting and cumbersome, yet it is believed that the student who does continue on to algebra will be capable of the small transition to the more formalized structure.

- It is suggested that a teacher make optimum use of a long-range and short-range preplanning for any particular unit being taught so that a variety of visual teaching aids may be constructed or obtained which might help students gain a better understanding of the topic being discussed.

- It is suggested that student proficiency and understanding be major aims of a teacher when these units are taught, with minor concern devoted

to how many of these units are covered. A time allotment is suggested as a guide, but teachers may alter this in either direction depending on the progress of their classes. Student interest and achievement may determine how much time is spent on each unit. The supplementary units may replace some of the basic units as the teacher feels is necessary.

2. Goals of the Course

- This course should provide meaningful mathematical experiences for the prealgebra students with emphasis on developing the concepts, skills, and techniques that will ensure their success in algebra.

- This course is basically for the prealgebra student, but with more emphasis on the supplementary units, it may be used with the terminal student.

3. Specific Comments Relative to Each Section

Unit I. Graphing - The topic "graphing" should be approached as a transition from work on the number line. An insight into what is meant by a linear relationship should be aimed at as a goal in terms of straight-line graphs. The construction of various types of graphs (bar graphs, pictographs, etc.) should not be a point of concentration, but ample time should be spent on critical analysis and interpretation of these graphs. Graphing linear relations is restricted to the first quadrant at this time.

Unit II. Number Bases - Number bases should be explored in a manner that will reinforce the concepts relative to the decimal system rather than attempt to develop computational skills in other bases.

Unit III. Set of Integers - The set of integers should be examined in such a manner that the need for negative numbers in our number system is well illustrated. Students should be made aware of the field axioms that are satisfied in the set of integers. A thorough examination of the operations of addition and subtraction should be presented. A heavy concentration of number phrases, open sentences, and the solution of number sentences, along with the usage of the addition and subtraction properties of equality, should be included in this section. Problem solving should be used to reinforce these concepts. Multiplication and division are deferred until Unit IV.

Unit IV. Rational Numbers - A thorough understanding of prime and composite numbers is essential at the very beginning of this unit in order that the operations of finding a greatest common factor and least common multiple can be established. After the rational numbers are developed, graphing is extended into all four quadrants.

Unit V. Metric Geometry - Formula sheets should be made available to students once the basic perimeter and area formulas have been developed. A strict methodology should be adhered to by students when using these formulas. Students should become proficient in the use of simple measuring instruments, such as the ruler, the compass, and the protractor. Some simple constructions are presented as part of this unit.

Unit VI. Ratio, Proportion, and Percent - Percentage problems should be solved through the use of a proportion.

Unit VII. Probability and Statistics - A better understanding of probability processes can be acquired by doing a few experiments of a simple nature, tabulating the results, and analyzing them. Emphasis should be placed on empirical probability.

Supplementary Units VIII - XIV - These units are provided to extend a student's ability to use mathematics from a practical standpoint, as well as to provide more points of contact with the mathematical world in general. Some of the units provide a review or extension of previously encountered concepts, while others introduce useful, interesting, and just plain fun topics from various branches of mathematics.

The units on informal geometry and volumes of geometric solids should not be presented before basic Unit V - Metric Geometry.

The unit on the slide rule should not be presented until students are familiar with operations dealing with decimal numbers. Thus, it is suggested that it follow somewhere after basic Unit IV - Rational Numbers.

The units, Recreational Mathematics and Consumer Mathematics, contain material not to be taught in regular unit style. Areas from them should be presented at different times throughout the year when you, as a teacher, feel it is most appropriate and in accord with student interest and involvement.

The unit, Flow Charts and Calculators, does not necessitate that every student in class have a calculator to work with, although this would be the ideal situation. If group work activities are used, this unit may be successfully completed with only a limited number of calculators. You will find that some flow chart work may be possible even in the absence of a calculator. This unit can be introduced anywhere in the course and be used throughout.

The unit, Mathematical Reasoning, tends to develop informal reasoning in the student. It is not intended as a rigorous course in logic.

INTRODUCTION TO HIGH SCHOOL MATHEMATICS
GRADE 9

Scope and Content

Basic Units (I - VII)	Time Allotment (Days)
I. Graphing	19-21
A. Points on a line	
1. Horizontal number line	
2. Vertical number line	
B. Points in a plane	
1. Combining the horizontal and vertical number lines	
2. Ordered pairs	
a. Game called "Battleship"	
3. Graphing ordered pairs	
a. Pictures	
b. Geometric figures	
C. Patterns in ordered pairs	
1. Lines	
D. Interpretation of common graphs	
E. Examining data and graphs	
F. Graphs and formulas	
G. Graphs of inequalities	
II. Number Bases	9-11
A. The decimal system	
1. The ten symbols of the decimal system	
2. Number and numeral	
3. The meaning of place value in expanded form	
4. The meaning of place value in exponential form	
B. Other systems of numeration	
1. The n symbols of base n	
2. Place value in other bases in exponential form	
3. Place value in other bases in expanded form	
4. Counting in other bases	
5. Addition and multiplication tables in other bases	

III. Set of Integers 18-24

- A. Need for and definition of integer
  - 1. Closure under subtraction
  - 2. Distributive property of multiplication over addition
  - 3. Order of operations
- B. Integers on the number line as directed numbers
- C. Addition on the number line
- D. The existence of an additive inverse for every integer with zero as its own additive inverse
- E. Subtraction on the number line
- F. Concept of order on the number line
  (1. Use of > and <, i.e., +8>+5; -5<-2)
- G. Use of a matrix (to reemphasize the rules of sign for addition and subtraction).
- H. Mathematical sentences
  - 1. Definition
  - 2. Truth or falsity of open sentences
  - 3. Replacement and solution sets
- I. Problem-solving techniques
  - 1. Forming the open sentence that is representative of the problem
  - 2. Finding the solution set for open sentences

IV. Rational Numbers 48-58

- A. Multiplication in the set of integers
- B. Division in the set of integers
- C. Need for and definition of rational number
- D. Rationals on the number line
- E. Order in the set of rationals
- F. Operations with rational numbers
  - 1. Prime and composite numbers
    - a. Definition of prime and composite
    - b. Test for primality
    - c. Prime factorization of composite numbers
    - d. Unique factorization theorem
  - 2. Techniques of finding the lowest common multiple and greatest common factor
  - 3. Addition

a. Relating the skills of L.C.M. and prime factorization to the addition operation
b. Transition from vertical to horizontal arrangement
c. Adding positive and negative rationals

4. Subtraction
a. Develop technique for subtraction of rationals
b. Reinforce concept that addition and subtraction are inverse operations

5. Multiplication
a. Definition a/b x c/d = a x c / b x d
b. Using prime factors and commutativity as aids in multiplication

6. Division
a. Definition of division and meaning of reciprocal a/b ÷ c/d = a/b x d/c
b. Rational numbers and their multiplicative inverses
c. Other division algorithms

7. Changing improper fractions to mixed numbers

G. More mathematical sentences
H. More graphing
I. Decimals
1. Negative powers of ten and their decimal equivalents
2. Fraction - decimal and mixed number - decimal conversions
3. Rational numbers as repeating decimals
4. The four operations with decimals

J. Problem solving
1. Brief review of those concepts previously developed in the section on integers
2. Finding solution sets for open sentences containing rational numbers
3. Solution of problems whose related open sentences contain noninteger rational numbers

V. Metric Geometry 18-22
A. Use of protractor in angle measurement
B. Constructions

C. Perimeter
   1. Square
   2. Rectangle
   3. Triangle
D. Area
   1. Square
   2. Rectangle
   3. Parallelogram
   4. Triangle
   5. Trapezoid
E. Circle
   1. Definition of circumference, diameter, and radius
   2. Develop formula for circumference
   3. Develop formula for area
F. Denominate numbers as they apply to linear and square measurement
G. Pythagorean theorem - drawing the squares on the sides and hypotenuse of a right triangle and computing the areas

VI. Ratio, Proportion, and Percent 14-16
A. Ratio
   1. Definition of ratio
   2. Difference between rate and ratio
   3. Writing ratios
   4. Reducing ratios
B. Proportion
   1. Definition of proportion
   2. Writing proportions
   3. The terms of a proportion
   4. Means and extremes
   5. The proportion as a special case of the open sentence
   6. The cross product rule
C. Percent
   1. Definition of percent
   2. Problems involving percent solved by proportion
D. Related activities

VII. Probability and Statistics 15-18
A. Probability
   1. Definition
   2. Empirical probability
   3. Mathematical probability

B. Statistics
   1. Organizing data by means of frequency tables and histograms
   2. Statistical measures
      a. Mean
      b. Mode
      c. Median

Supplementary Units (VIII - XIV)

VIII. Informal Geometry
   A. Point, line, plane
      1. Point
         a. Undefined nature of point
      2. Line
         a. Undefined nature of line
         b. Infinity of points on a line
         c. Uniqueness of a line determined by two points
         d. Intersecting lines
         e. Parallel lines
      3. Plane
         a. Undefined nature of a plane
         b. Three noncollinear points determine a plane
         c. Intersection of a line and a plane
         d. Intersection of two planes
   B. Triangles and Polygons
      1. Triangles
         a. Definition of triangle
         b. Types of triangles
      2. Polygons
         a. Definition of polygon
         b. Convex and concave polygons
         c. The closed curve and its interior and exterior regions
         d. Naming the polygons
   C. Figures of three dimensions
      1. Definition of space
      2. Relationships of points, lines, and planes in space
      3. Some common three-dimensional figures

IX. Use of the slide rule
- A. Examination of C and D scales
- B. Multiplication on the slide rule
  1. Basic procedure
  2. Location of decimal point in the final product
  3. Estimating the location of a terminal digit in the multiplication process
- C. Division on the slide rule
- D. Finding the square of a number on the slide rule
- E. Finding square roots of numbers on the slide rule

X. Recreational Mathematics
- A. Prime numbers
- B. Magic squares
- C. Paper cutting
- D. Palindrome numbers
- E. Cross number puzzles
- F. Number puzzles
- G. Coordinate pictures
- H. Building numbers
- I. Curve stitching
- J. Cutting a cube

XI. Flow Charts and Calculators
- A. General discussion
- B. Examples of flow charts

XII. Volumes of Geometric Solids
- A. The concept of volume
  1. Development of a unit cube
  2. Use of the unit cube in the development of formulas
- B. The general formula $V = Bh$
  1. Use of models and other aids to show relationship between the volumes of various geometric solids
  2. The construction of various models
- C. General suggestions to teachers

XIII. Consumer Mathematics
- A. Consumer mathematics as an outgrowth and enlargement of areas of student interest
- B. Some examples of topics for discussion

XIV. Mathematical Reasoning
- A. Simple sentences
  - 1. Quantifiers
- B. Compound sentences
  - 1. Connectives
  - 2. Symbols
- C. Venn diagrams
- D. Truth tables
- E. Applications

INTRODUCTION TO HIGH SCHOOL MATHEMATICS

GRADE 9

Suggestions to Teachers

UNIT I. Graphing

Note: It may be advisable to begin by reviewing some concepts of sets, keeping in mind that the students should have had this in previous years.

Ex: Natural numbers, whole numbers, integers, rationals

A. Sets of points on a line - Review the concept of points on a line, number line and directed number. Discuss the arbitrary positioning of the number line, i.e., horizontal or vertical positioning.

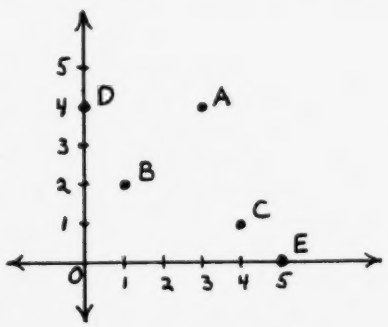
B. Sets of points in a plane - The concept of designating the position of a point in a plane can be approached by combining the vertical and horizontal number lines so as to produce a proper frame of reference. These lines are most often referred to as axes. Points on the two number lines can now be designated as 3 units to the right of 0 on the horizontal line, or 2 units below 0 on the vertical line. To develop the idea of an ordered pair, a discussion of the point of intersection of the two number lines as being 0 on the horizontal line and 0 on the vertical line can lead to the idea of its being represented by the ordered pair (0,0). This can be extended to include other points on the horizontal line as (3,0); (-7,0), etc., and points on the vertical line as (0,5); (0,-3), etc. A discussion of why all points on the horizontal line have 0 as the second element of their ordered pairs and all points on the vertical line have 0 as the first element of their ordered pairs can lead to an easy transition to the ordered pair for any point in the plane. The student should now develop the activity to graph ordered pairs, i.e., given ordered pairs, find corresponding points and given points, find corresponding ordered pairs.

Two appropriate activities at this time are the game of "Battleship" and Picture Graphs. One version of this game can be found in the Bureau of Secondary Curriculum Development's handbook, Measurement - a resource unit for a course

in Basic Mathematics, p. 23. The version below locates the ships on lattice points rather than squares.

In "Battleship," each player has a certain number of ships, located on a coordinate system unseen by his opponent. The object of the game is to "sink" the opponent's ships by guessing their location using coordinates. The players alternate "firing" until all the opponent's ships are sunk. Each player should record his "shots" to avoid repetition.

Example of one player's ships in "Battleship"

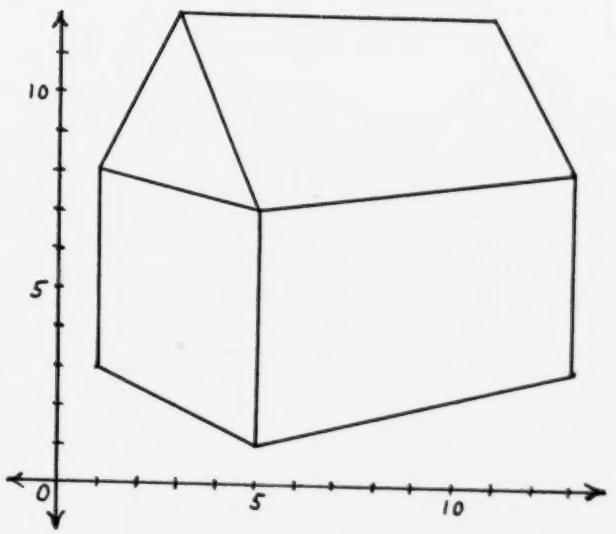


Give the students sets of points that form pictures such as of a rabbit, boat, house, etc., when connected in succession. Also give points that form geometric figures, i.e., square, rectangle, triangle.

Example: Locate the following points and connect each point, in each group, to the succeeding point in the order listed.

a) (1,8), (3,12), (5,7), (13,8), (13,3), (5,1), (5,7), (1,8), (1,3), (5,1)

b) (3,12), (11,12), (13,8)



C. Patterns in ordered pairs - Students should learn to recognize ordered pairs that have the same relationship and be able to find other ordered pairs with the same relationship.

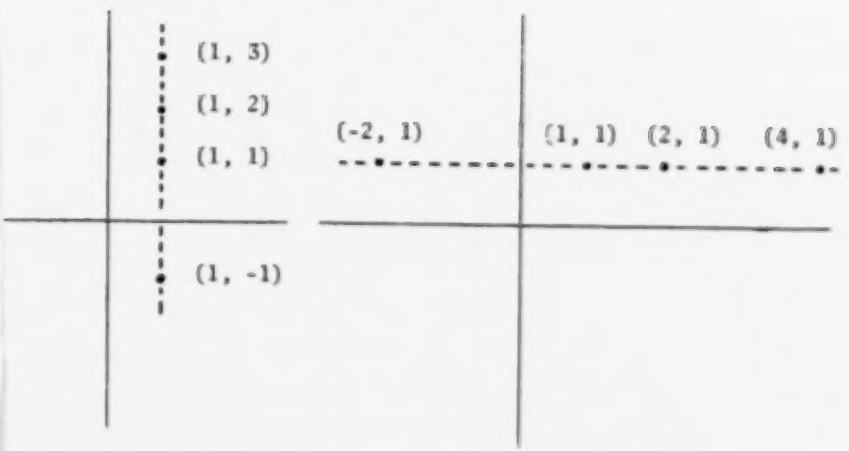
Example:

Given (3,7), (6,10), (11,15), find (12,?) and (?,17)

Graphing the three given ordered pairs of the above example and connecting them will provide students with some insight into what is meant by linear relationship; graphing the two ordered pairs they found will provide a needed reinforcement of this concept.

Given a line parallel to the horizontal number line, select any four points and determine their corresponding ordered pairs and compare the second numbers of the four ordered pairs. Repeat for lines parallel to the vertical number line, and compare the first numbers of the ordered pairs.

Example:



There should be enough examples to emphasize and reinforce the recognition that lines parallel to the vertical number line always have the first number of the ordered pair constant and lines parallel to the horizontal number line always have the second number of the ordered pair constant.

D. The interpretation of a graph - This can be approached by first examining the title or description of a graph in an attempt to get the general idea of what is desired to be conveyed to the reader. An examination of the scales that are being used, as well as the intervals, will help to bring into focus the method in which the total graph is being displayed. Some time should be spent extracting specific bits of information from the graph and discussing the pertinent facts related by the graph.

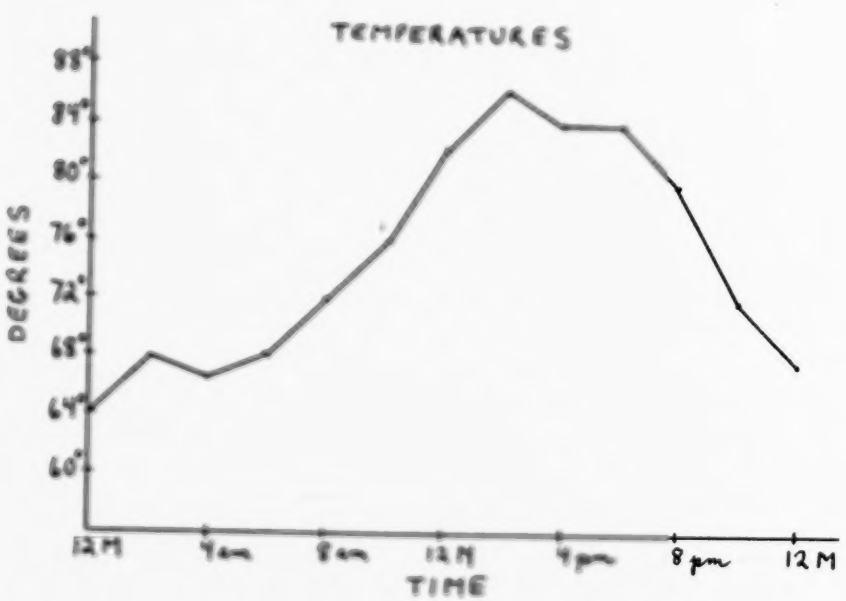
Multiple-line graphs, band graphs, and scatter diagrams provide good examples of graphing problems where both the interpretation of a graph and comparative information from the graph itself may be discussed.

The chosen scales of graphs should be examined in terms of the intervals chosen and the visual effect produced by enlarging or reducing the size of these intervals. Newspapers and magazines oftentimes provide graphs, where intervals have been chosen such that a particular visual outcome is displayed in relation to the data being considered.

As a reference to the material being discussed here, and for further suggestions and creative ideas, consult the Bureau of Secondary Curriculum Development's handbook, *Graphs and Statistics*.

E. Examining data and graphs - A graph is a visual portrayal of data. The main objective is to be able to read and interpret each kind, to apply the results to problem situations, and finally, to construct each variety.

The bar graph consists of vertical or horizontal bars of varying length to illustrate discrete data. The line graph is a broken line designed to display continuous data, plotting one variable against a second. For example, temperature recorded at hourly intervals lends itself to a line interpretation, while automobile production in several countries can be shown best on a bar graph. On the basis of the line graph, certain statistical concepts can be illustrated.



Questions to be asked:

What was the hottest temperature recorded, and at what time did it occur? (86°; 2 p.m.)

During what 2-hour period was the temperature most nearly constant? (4 p.m. to 6 p.m.)

At what two times of the day was the temperature 76°? (10 a.m. and 9 p.m.)

A great deal of emphasis should be placed on the determination of proper scales to be used in the graphing of a particular problem. An examination of the range of the data to be graphed should provide the student with a more basic understanding of the actual problem and also should lead to more insight in the interpretation of a graph. The analysis of a number of problems dealing with the selection of a good scale is suggested before the student is asked to proceed with the activities involved in the construction and completion of a graphing problem.

It is suggested that some experience be given to the student that would involve the graphing of a curved line. This would serve two purposes: (1) to remove any assumption the student might now have that all graphs are composed of points connected by straight lines, and (2) to provide the student with an opportunity to make implicative predictions beyond what is visually graphed. Consult the Bureau of Secondary Curriculum Development's handbook, *Graphs and Statistics*, p. 35.

F. Graphs and formulas - Graphs are very useful in displaying number relationships. Although it is not expected students be given formal and rigid definitions, the following vocabulary is suggested: mathematical sentence, set of points, variables, solution set, satisfies.

1. Formulas

   a. If pretzel sticks cost 3¢ each, the relation may be written as:

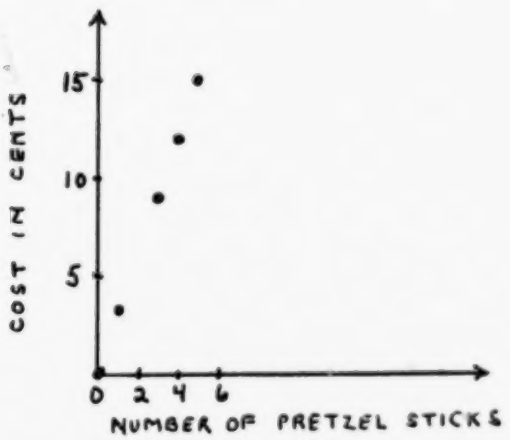
$$C = 3 \times P$$

$$C = 3 \cdot P$$

$$C = 3P$$

b. Table of values

P	4	1	0	3	5
C	12	3	0	9	15



The graph consists of a set of points. The student should not draw the line but should be able to see that the pattern is a line. The concept of a set of points instead of a line should be emphasized.

c. Formulas of a general nature.

$y = 5x$ $y = x + 3$

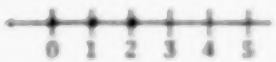
1. table of values

2. graph - after the student sees the pattern for a line, he should draw the line.

Note: In the pretzel problem, the value for P is restricted to whole numbers, whereas in the general graph $y = 5x$, x can take on fractional values.

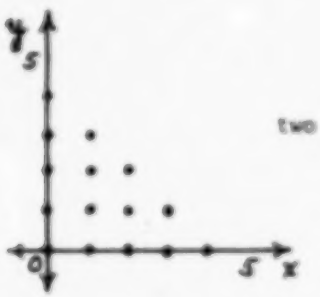
C. Graphing inequalities involving whole numbers

1. $x < 3$



one dimensional

2. $x + y < 5$



two dimensional

UNIT II. Number Bases

A. The decimal system - It is useful to initiate a class discussion of the history and development of the decimal system with some emphasis on the differentiation between number and numeral, the ten symbols of the decimal system, and of their arrangements that yield the numerals of the decimal system. Place value should be discussed. This discussion should lead to the relationship that $342 = (3 \times 100) + (4 \times 10) + (2 \times 1)$. Depending on the individual student's background, it may be necessary to discuss the concept of exponents in order that each student will develop the skill of expressing place value in exponential form. It is important here that students understand that $10^0 = 1$ and further that $N^0 = 1$ for $N \neq 0$. The goal of the above proceedings is that a student will develop the skills of expressing a number in each of the following ways:

$$652 = (6 \times 100) + (5 \times 10) + (2 \times 1)$$
$$= (6 \times 10^2) + (5 \times 10^1) + (2 \times 10^0)$$

B. Other systems of numeration - An extension of the fact that the decimal system (base 10) has 10 symbols would be to show that in base 5 there are 5 symbols, in base 7

there are 7 symbols, and that the generalization of this idea leads to the conclusion that in base n there would be n symbols.

Emphasize the fact that the units place will now be known as the zero power of the base place, and that successive places to the left will have values determined by successive power of the base. For example,
$342_{five} = (3 \times 5^2) + (4 \times 5^1) + (2 \times 5^0)$.

Students should learn to count in other bases.

The skill of constructing addition and multiplication tables for other systems of numeration should be developed. A suggestion, as an aid to the construction of these tables, would be to set up a correspondence between the base 10 numbers and the numbers in the base with which the table is concerned. For example, in base 4:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	10
2	2	3	10	11
3	3	10	11	12

base 10	base 4
1	1
2	2
3	3
4	10
5	11
6	12
7	13
8	20
9	21

x	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	10	12
3	0	3	12	21

base 10	base 4
1	1
2	2
3	3
4	10
5	11
6	12
7	13
8	20
9	21

UNIT III. Set of Integers

A. The properties of addition and multiplication in the set of whole numbers - These properties should be reviewed in class discussion with particular emphasis on the closure property of whole numbers under these two operations. Discuss the failure of the set of whole numbers to close under subtraction and, hence, the need for the opposites or negatives of the natural numbers. The student should understand that unlike addition and multiplication, subtraction is not commutative nor does the associative property apply. The distributive property should be emphasized. Order of operations should be introduced using parentheses, multiplication, division, addition, and subtraction.

B. The number line for the integers - This may be developed from the number line for whole numbers in class discussion through which students will become aware that the negative integers are measured from zero in a direction opposite that of the positive integers. In this way, the concept of directed numbers may be developed.

C. Addition on the number line - Examples should be approached slowly so students will come to understand each of the variety of possibilities that can occur.

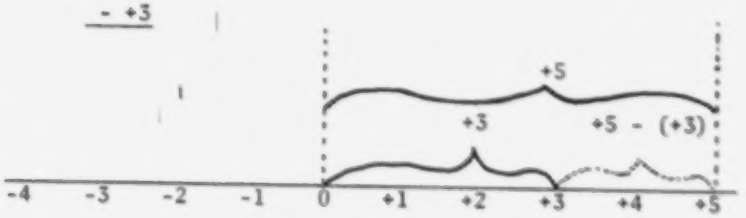
   For example, the sum of two positive integers can be introduced first. Here, it might be well to emphasize that the addition of a positive integer will mean movement to the right on the number line. A number of examples showing the addition of two positive integers on the number line will lead to the realization that the sum will be positive. In a similar fashion, the addition of two negative integers will produce a sum that is negative. It should be emphasized that the addition of a negative integer will mean movement to the left along the number line. The process of adding a negative integer to a positive integer can now be easily accomplished. It must be emphasized that such a sum will be positive when the integer having the greatest absolute value is positive; and negative, when the integer having the greatest absolute value is negative.

D. The existence of an additive inverse - Students should become aware that every integer has an opposite or additive inverse,

with zero being its own additive inverse. Any integer added to its additive inverse will produce zero. The identity property of zero with respect to addition can be illustrated by an example.

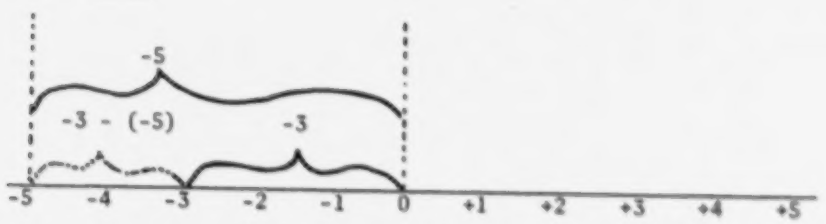
E. Subtraction on the number line - Subtraction should be interpreted as the difference on the number line after both integers have been set off from zero with the direction or sign being read from the subtrahend to the minuend. Emphasize again that direction to the right is positive and direction to the left is negative.

Ex: +5
- +3



What must be subtracted from +5 to get +3?
2 intervals read to the right, hence +2

Ex: -5
- -3



What must be added to -3 to get -5?
2 intervals read to the left, hence -2

From this a rule should be developed for the subtraction of signed numbers.

F. The concept of order on the number line - This concept is important, and students should understand that for any two distinct integers on the number line, the integer to the right is the greater and the integer to the left is the lesser. This concept can be combined with the concept of inequality in simple illustrative examples.

$$+8 > +5$$
$$-5 < -2$$

G. The use of a matrix for addition and subtraction of integers - This can be very useful in reinforcing many concepts previously developed. The following matrices can be easily developed in class with the students by using ascending and descending number patterns. Examination of the matrices will verify the rules of sign previously developed as well as the property of zero and additive inverse.

+	+4	+3	+2	+1
+4	+8	+7	+6	+5
+3	+7	+6	+5	+4
+2	+6	+5	+4	+3
+1	+5	+4	+3	+2

0	+4	+3	+2	+1
0	+4	+3	+2	+1
-1	+3	+2	+1	0
-2	+2	+1	0	-1
-3	+1	0	-1	-2
-4	0	-1	-2	-3

0	-1	-2	-3	-4
+4	+3	+2	+1	0
+3	+2	+1	0	-1
+2	+1	0	-1	-2
+1	0	-1	-2	-3

0	-1	-2	-3	-4
0	-1	-2	-3	-4
-1	-2	-3	-4	-5
-2	-3	-4	-5	-6
-3	-4	-5	-6	-7
-4	-5	-6	-7	-8

-	+4	+3	+2	+1	0	-1	-2	-3	-4
+4	0	+1	+2	+3	+4	+5	+6	+7	+8
+3	-1	0	+1	+2	+3	+4	+5	+6	+7
+2	-2	-1	0	+1	+2	+3	+4	+5	+6
+1	-3	-2	-1	0	+1	+2	+3	+4	+5

0	+4	+3	+2	+1	0	-1	-2	-3	-4
-1	-5	-4	-3	-2	-1	0	+1	+2	+3
-2	-6	-5	-4	-3	-2	-1	0	+1	+2
-3	-7	-6	-5	-4	-3	-2	-1	0	+1
-4	-8	-7	-6	-5	-4	-3	-2	-1	0

H. Mathematical sentences

1. There should be some discussion regarding the analogy between mathematical sentences and English sentences, with emphasis on the part that the signs =, >, and < play.

   Examples:

   If 6 is subtracted from 11, the result is 5.

   $$11 - 6 = 5$$

   If 7 is added to 9, the result is greater than 15.

   $$9 + 7 > 15$$

   If -8 is added to +6 the result is less than zero.

   $$+6 + (-8) < 0$$

   Students should come to realize that sometimes arbitrary place holders for numbers in mathematical sentences may be used.

Example:

A certain integer plus seven equals twelve.

$? + 7 = 12 \quad \therefore ? = 5$

$\square + 7 = 12 \quad \therefore \square = 5$

$x + 7 = 12 \quad \therefore x = 5$

While ?, □, and x are all suitable place holders for numbers in mathematical sentences, students should be encouraged to use the letter form of a place holder.

2. The concepts of open, true, and false mathematical sentences can be approached through the use of English sentences.

Example:

He was a President of the United States. (T or F, open to question.)

George Washington was a President of the United States. (Statement is true.)

Babe Ruth was a President of the United States. (Statement is false.)

In similar fashion:

$x + 3 = 7$ (T or F, open to question)

$x + 3 = 7$ and $x = 4$. (Statement is true.)

$x + 3 = 7$ and $x \neq 4$. (Statement is false.)

3. Students should be introduced to the ideas of what constitutes an acceptable replacement for a place holder for a number in a mathematical sentence and that solution set means all those numbers that make the statement true.

I. Problem solving

1. In discussing problem solving, students should be made aware that the most powerful method of problem solving involves the development of an open sentence that is representative of the problem and then finding the solution set for that open sentence. There are, of course, many related skills that students must acquire before this technique can be successfully applied. A basic skill is that of translating English phrases into mathematical phrases. In such statements as "a number increased by 7," the use of ?, □, or x for a number is arbitrary, but the use of the letter form should be encouraged. After the skill of translation has been developed, a good deal of time will need to be spent on the analysis of problems. As a result, the various mathematical phrases needed to formulate an open sentence that is representative of a problem may be determined more easily.

Example:

Mary is 12 years older than John. If Mary is 17 years old, how old is John?

"How old is John" implies that John's age is x. "Mary is 12 years older than John" implies that Mary's age is $x + 12$.

Mary's age is also given as 17 and, therefore, $x + 12 = 17$.

Many simple problems of this type will need to be explored so that students may have ample opportunity to develop the skills of translation, analysis, and open sentence formulation.

2. The process of finding a solution set should be approached slowly. Very simple sentences involving only the addition and subtraction axioms should be used in the development of the elementary concepts of solving an equation. Students will proceed to solve such equations by an intuitive process often referred to as inspection. The students will need to be brought through a transition from this intuitive approach to a more formalized methodology. They will probably strongly resist this

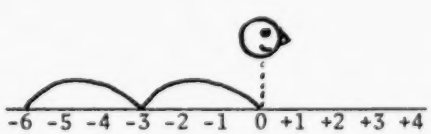
transition. For example, in the equation $x + 5 = 7$, a student can easily be made to realize that he is looking for a number such that when 5 is added to it, the sum will be 7. He will be able to find the solution, 2, by an intuitive process which he will be unable to explain. He will probably resist the attempt to formalize by applying the axiom of subtraction. This resistance may be overcome by presenting to the student an equation such as $x + 97 = 243$, and pointing out the fact that the intuitive approach is difficult and that the axiom of subtraction makes the solution comparatively easy. It should be pointed out that the solution of a mathematical sentence is not complete until the solution set has been tested in the original sentence, i.e., check.

UNIT IV. Rational Numbers

A. Multiplication in the set of integers - Quite frequently the use of arbitrary devices can be helpful in developing difficult concepts. Such is the case when we consider multiplication on the number line. For example, we may consider multiplication as a succession of jumps on the number line in which the sign of the multiplicand represents an initial facing direction starting at zero, the magnitude of the multiplicand represents the size of each jump interval, the sign of the multiplier represents either a forward or reverse direction, and the magnitude of the multiplier represents the number of jumps. Using this device, products and their corresponding signs are easily formed on the number line.

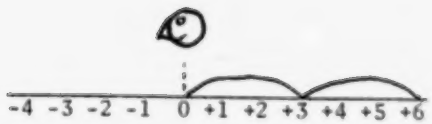
Ex: $+3$
$\times -2$

Initial facing:
to the right
Interval of jump:
3 units
Number of jumps: 2
Direction of jump:
reverse
Product: -6



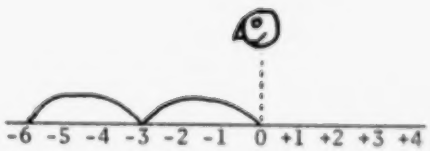
Ex: -3
x -2

Initial facing:
to the left
Interval of jump:
3 units
Number of jumps: 2
Direction of jump:
reverse
Product: +6



Ex: -3
x +2

Initial facing:
to the left
Interval of jump:
3 units
Number of jumps: 2
Direction of jump:
forward
Product: -6



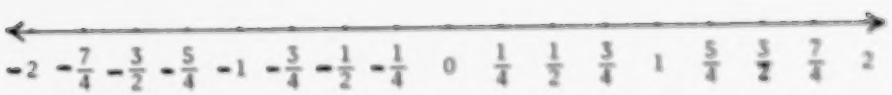
B. Division in the set of integers - The teacher should develop the rules for dividing signed numbers.

C. Need for rational numbers - Review the concept that subtraction was not always possible in the set of whole numbers and, consequently, the opposites of all the natural numbers were added to form the set of integers so that subtraction would always be possible. It is important that students recognize and understand that just as subtraction is not always possible in the set of whole numbers, division is not always possible in the set of integers, and, therefore, a need has been created for those numbers usually referred to as fractions.

Example: $2 \div 3 = \frac{2}{3}$ $7 \div 4 = \frac{7}{4}$

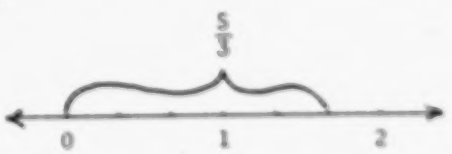
D. Rationals on the number line - A teacher-directed class activity aimed at extending the number line to include the rationals is the next logical development.

Example:

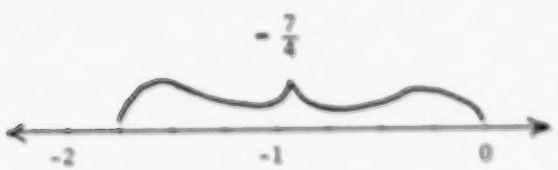


A reinforcement of the fraction concept can be achieved by some examples showing the representation on the number line of some simple fractions.

Example: $\frac{5}{3}$



Example: $-\frac{7}{4}$



Point out that every integer is also rational.

E. Order in the set of rationals - The set of rationals can be ordered for the same reason as the set of integers.

F. Operations with rational numbers

1. The operations with rational numbers can best be approached after the student has reached a thorough understanding of prime and composite integers and the concept of least common multiple and greatest common factor. The students should realize that a prime number is an integer greater than 1 that is divisible without remainder by itself and 1 only. Further, it should be clear that an integer that is not prime or 1 is composite and has factors other than 1 and itself.

Example:	Prime No.	Factors Without Remainder
	7	7,1
	3	3,1
	13	13,1

	Composite No.	Factors Without Remainder
	9	1,3,9
	8	1,2,4,8
	12	1,2,3,4,6,12

Students should also become aware that each composite number can be divided without remainder by at least one of the primes not greater than the square root of the number itself (approximate square root by estimation - probably some practice will be needed with this procedure).

Example:	Prime No.	Prime Factors
	12	2,3
	16	2
	28	7,2
	35	7,5

Students should also come to understand that if a number cannot be divided without remainder by any of the primes not more than its square root, then that number must be prime.

Example:

Prime No.	App. Sq. Root	Test Divisibility by Primes
61	< 8	2,3,5,7
97	< 10	2,3,5,7
127	< 12	2,3,5,7,11
367	< 20	2,3,5,7,11,13,17,19

The task of finding the prime factors of a composite number can be simplified to some extent by an understanding of the rules for divisibility by 2,3,5, i.e.,

Every even number (numbers ending in 0, 2, 4, 6, or 8) is divisible without remainder by 2.

Every number for which it is true that the sum of its digits is divisible without remainder by 3 can itself be divided without remainder by 3.

Every number whose last digit is 0 or 5 can be divided without remainder by 5.

Example: Find prime factors of 570.

570 ends in 0 and hence has a divisor of 5	
5 x 114	114 is even and has a divisor of 2
5 x 2 x 57	57 has digits whose sum is 12 and hence a divisor of 3
5 x 2 x 3 x 19	19 is prime

The student should come to realize that the prime factorization of each composite number is unique. The order of the factors may vary.

Example:

$120 = 2 \times 60 = 2 \times 4 \times 15 = 2 \times 2 \times 2 \times 15 = 2 \times 2 \times 2 \times 5 \times 3$
$120 = 4 \times 30 = 4 \times 5 \times 6 = 2 \times 2 \times 5 \times 6 = 2 \times 2 \times 5 \times 3 \times 2$
$120 = 15 \times 8 = 5 \times 3 \times 8 = 5 \times 3 \times 2 \times 4 = 5 \times 3 \times 2 \times 2 \times 2$

Some applications of the above skills and concepts can be found in the following: Reducing fractions by means of prime factors.

Example:

$$\frac{196}{24} = \frac{2 \times 2 \times 7 \times 7}{2 \times 2 \times 2 \times 3} = \frac{7 \times 7}{2 \times 3} = \frac{49}{6}$$

At this point the student should be provided with a list of the prime numbers less than 30 to save time in performing operations calling for these numbers.

2. Find a greatest common factor for a sequence of numbers by using prime factors.

Example: Find G.C.F. for: 12, 16, 40

$12 = 2 \cdot 2 \cdot 3$
$16 = 2 \cdot 2 \cdot 2 \cdot 2$
$40 = 2 \cdot 2 \cdot 2 \cdot 5$
$\text{G.C.F.} = 2 \cdot 2 = 4$

Note that the G.C.F. is formed by taking only those prime factors that appear in every factorization and using each such prime factor the least number of times that it appears in any single factorization.

Finding a least common multiple of a sequence of numbers using prime factors.

Example: Find L.C.M. for: 8, 12, 15

$$8 = 2 \cdot 2 \cdot$$
$$12 = 2 \cdot 2 \cdot$$
$$15 = 5 \cdot 3$$
$$\text{L.C.M.} = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 = 120$$

Note that the L.C.M. is formed by taking each prime factor the greatest number of times that it appears in any single factorization and then finding the product.

It should be emphasized to the students that L.C.M. of the denominators of a series of fractions is frequently referred to as the lowest common denominator.

3. Addition of rational numbers

   a. The student should be made aware that the concepts and skills related to finding a L.C.M. are used to great advantage when one performs operations with rational numbers. When adding rational numbers, he should know that keeping the lowest common denominator in its prime factored form will facilitate the process of addition. There should be a transitional stage between the vertical arrangement of the work and the horizontal arrangement.

   b. Transitional vertical arrangement

Example:
$$\frac{3}{12} = \frac{3}{2\cdot2\cdot3} = \frac{?}{2\cdot2\cdot2\cdot3} = \frac{6}{2\cdot2\cdot2\cdot3} = \frac{6}{24}$$
$$\frac{5}{8} = \frac{5}{2\cdot2\cdot2} = \frac{?}{2\cdot2\cdot2\cdot3} = \frac{15}{2\cdot2\cdot2\cdot3} = \frac{15}{24}$$
$$+\frac{2}{3} = \frac{2}{3} = \frac{?}{2\cdot2\cdot2\cdot3} = \frac{16}{2\cdot2\cdot2\cdot3} = +\frac{16}{24}$$
$$\frac{37}{24}$$

horizontal arrangement

1) $= \frac{3}{12} + \frac{5}{8} + \frac{2}{3}$

2) $= \frac{3}{2\cdot2\cdot3} + \frac{5}{2\cdot2\cdot2} + \frac{2}{3}$

3) $= \frac{?}{2\cdot2\cdot2\cdot3} + \frac{?}{2\cdot2\cdot2\cdot3} + \frac{?}{2\cdot2\cdot2\cdot3}$

4) $= \frac{6 + 15 + 16}{2\cdot2\cdot2\cdot3}$

5) $= \frac{37}{24}$

Note that in the vertical arrangement above prior skills and concepts are combined with the technique of using the prime factored form of L.C.D.

Enough time should be spent on the transition to a vertical arrangement so that the student will master this combination of old and new concepts and skills. In developing the horizontal arrangement, both arrangements should be drawn simultaneously, and the relationships between them should be clearly pointed out to the student.

c. A strong emphasis needs to be placed on the technique of adding positive and negative rational numbers.

Examples:

$= \frac{4}{14} + \left(-\frac{3}{8}\right) + \left(\frac{-7}{15}\right)$

$= \frac{4}{2\cdot7} + \left(\frac{-3}{2\cdot2\cdot2}\right) + \left(\frac{-7}{3\cdot5}\right)$

$= \frac{?}{2\cdot2\cdot2\cdot3\cdot5\cdot7} + \left(\frac{-?}{2\cdot2\cdot2\cdot3\cdot5\cdot7}\right) + \left(\frac{-?}{2\cdot2\cdot2\cdot3\cdot5\cdot7}\right)$

$= \frac{240}{2\cdot2\cdot2\cdot3\cdot5\cdot7} + \left(\frac{-315}{2\cdot2\cdot2\cdot3\cdot5\cdot7}\right) + \left(\frac{-392}{2\cdot2\cdot2\cdot3\cdot5\cdot7}\right)$

$= \frac{240 + (-315) + (-392)}{2\cdot2\cdot2\cdot3\cdot5\cdot7}$

$= \frac{-467}{2\cdot2\cdot2\cdot3\cdot5\cdot7} = \frac{-467}{840}$

4. Subtraction of rational numbers

a. The operation of subtraction of rational numbers should be an easy development since no new skills or concepts are required. Only those skills and concepts learned in the addition of rational numbers and in the subtraction of integers are necessary. The teacher should be alert for any difficulties in the above areas and should be prepared to review and reemphasize any of these areas.

Examples:

① $\frac{3}{4} - \frac{1}{5}$

$= \frac{3}{2\cdot 2} - \frac{1}{5}$

$= \frac{?}{2\cdot 2\cdot 5} - \frac{?}{2\cdot 2\cdot 5}$

$= \frac{15}{2\cdot 2\cdot 5} - \frac{4}{2\cdot 2\cdot 5}$

$= \frac{15-4}{2\cdot 2\cdot 5}$

$= \frac{11}{20}$

② $\frac{1}{6} - \frac{7}{8}$

$= \frac{1}{2\cdot 3} - \frac{7}{2\cdot 2\cdot 2}$

$= \frac{?}{2\cdot 2\cdot 2\cdot 3} - \frac{?}{2\cdot 2\cdot 2\cdot 3}$

$= \frac{4}{2\cdot 2\cdot 2\cdot 3} - \frac{21}{2\cdot 2\cdot 2\cdot 3}$

$= \frac{4 - 21}{2\cdot 2\cdot 2\cdot 3}$

$= \frac{-17}{24}$

b. Reinforcement of those skills and concepts relative to addition and subtraction of rational numbers and the gaining of additional insights into this area might well be achieved by student participation in solution of pairs of problems as follows:

$\frac{2}{3} + \left(-\frac{3}{4}\right)$	$\frac{2}{3} - \frac{3}{4}$
$= \frac{2}{3} + \left(\frac{-3}{2\cdot 2}\right)$	$= \frac{2}{3} - \frac{3}{2\cdot 2}$
$= \frac{?}{2\cdot 2\cdot 3} + \left(\frac{-?}{2\cdot 2\cdot 3}\right)$	$= \frac{?}{2\cdot 2\cdot 3} - \frac{?}{2\cdot 2\cdot 3}$

$$= \frac{8 + (-9)}{2 \cdot 2 \cdot 3} \qquad = \frac{8 - 9}{2 \cdot 2 \cdot 3}$$

$$= \frac{-1}{12} \qquad = \frac{-1}{12}$$

5. Multiplication of rational numbers

   a. With respect to the operation, multiplication of rational numbers, the students should clearly understand that if any two rational numbers are multiplied, their product will always be a rational number. The student should realize that numbers like 2, 5, and 17 are also rational since they can be expressed as fractions in many ways.

Example:

$$2 = \frac{2}{1} = \frac{24}{12} = \frac{100}{50}$$

$$5 = \frac{5}{1} = \frac{25}{5} = \frac{100}{20}$$

$$17 = \frac{17}{1} = \frac{51}{3} = \frac{85}{5}$$

It should be emphasized quite clearly that the product of two or more fractions will be a fraction whose numerator is the product of the numerators and whose denominator is the product of the denominators.

   b. In order to reinforce previous skills and concepts and also facilitate the multiplication technique, the use of prime factors should be encouraged.

Example: $\frac{3}{16} \times \frac{12}{27} = \frac{3 \times 12}{16 \times 27} = \frac{3 \times 3 \times 2 \times 2}{3 \times 3 \times 3 \times 2 \times 2 \times 2 \times 2} = \frac{1}{12}$

There will be a need to overcome student resistance to the use of the prime factor method of multiplying rational numbers since techniques learned earlier may appear easier to use. This resistance must be overcome by careful explanation and demonstration that the prime factor method is the more powerful method of multiplying rational numbers.

Example: $\frac{77}{143} \times \frac{77}{154} = \frac{11 \times 7 \times 11 \times 7}{11 \times 13 \times 2 \times 7 \times 11} = \frac{7}{26}$

Emphasize also that the order in which two or more rational numbers are multiplied will not affect the product.

Example: $\frac{3}{4} \times \frac{4}{5} \times \frac{1}{2} \times \frac{2}{3} = \frac{1}{5}$

$\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} = \frac{1}{5}$

(You may find it interesting to discuss with the students the number of ways in which the above problem might be written by commuting the factors. Indeed, investigating examples of 2, 3, 4, 5, and 6 factors in a like manner can frequently produce some valuable mathematical insights and may lead the students into a sound generalization.)

6. Division of rational numbers

   a. Division of rational numbers causes some difficulty which may be overcome by bringing about an awareness on the part of the student that division is the inverse operation to multiplication, and that dividing by a rational number is equivalent to multiplication by the reciprocal of that rational number. Progress should be developmental so that a full understanding of the concept and of the meaning of reciprocal becomes clear to the student.

Example: $4 \div 2 = 2$ $\qquad$ $4 \times \frac{1}{2} = 2$

$8 \div 3 = 2\frac{2}{3}$ $\qquad$ $8 \times \frac{1}{3} = 2\frac{2}{3}$

If the concept of division of rational numbers is clear and the meaning of reciprocal is clear, then the transition to the following type of example should not be difficult:

$4 \div \frac{1}{2}$ must mean $4 \times \frac{2}{1}$ and therefore, 8

$8 \div \frac{1}{3}$ must mean $8 \times \frac{3}{1}$ and therefore, 24

$\frac{11}{2} \div \frac{7}{4}$ must mean $\frac{11}{2} \times \frac{4}{7}$ and therefore, $\frac{22}{7}$

b. Through discussion, the student should come to understand that every rational number, except zero, has a multiplicative inverse. The undefinable nature of the reciprocal of zero should be discussed and reciprocals of whole numbers should be discussed.

c. The teacher may wish to show other algorithms of division.

1. $\frac{3}{4} \div \frac{2}{5} =$

$$\frac{\frac{3}{4}}{\frac{2}{5}} \times \frac{20}{20} = \frac{15}{8}$$

or

$$\frac{3}{4} \div \frac{2}{5} =$$

$$\frac{15}{20} \div \frac{8}{20} = \frac{\frac{15}{8}}{1} = \frac{15}{8}$$

2. $\frac{3}{4} \div \frac{2}{5} =$

$$\frac{\frac{3}{4} \times \frac{5}{2}}{\frac{2}{5} \times \frac{5}{2}} = \frac{\frac{15}{8}}{1} = \frac{15}{8}$$

or

$$\frac{3}{4} \div \frac{2}{5} =$$

$$(\frac{3}{4} \times \frac{5}{2}) \div (\frac{2}{5} \times \frac{5}{2})$$

$$\frac{15}{8} \div 1 = \frac{15}{8}$$

7. Changing improper fractions to mixed numbers and mixed numbers to improper fractions.

G. More mathematical sentences

Review equations already solved. Now, it is appropriate to introduce and develop the solving of many different types of equations.

Example:

$3x = 20,\ \frac{x}{4} = 7,\ 3x + 7 = 16,$

$6x - 8 = 22,\ \ 3x + 4 = x + 10,\ \ 2(x + 10) = 28$

Although mention should be made of like terms, it should not be belabored. Checking should also be emphasized.

H. More graphing of linear equations

Now that the set of rationals has been developed, graphing can be extended into all four quadrants.

I. Decimals

1. Review from number bases the exponential form of place value, and extend to negative exponents. Perhaps the pattern approach can be helpful in developing the negative exponent concept.

Example:

$$10{,}000 = 10^4$$
$$1{,}000 = 10^3$$
$$100 = 10^2$$
$$10 = 10^1$$
$$1 = 10^0$$
$$\frac{1}{10} = 10^{-1}$$
$$\frac{1}{100} = 10^{-2}$$
$$\frac{1}{1000} = 10^{-3}$$
$$\frac{1}{10000} = 10^{-4}$$

The following transition can then be easily achieved:

$$1/10 = .1 = 10^{-1}$$
$$1/100 = .01 = 10^{-2}$$
$$1/1000 = .001 = 10^{-3}$$

Extension and reinforcement of the previous concepts should be possible by sufficient work on examples of the following type.

Example: Find the exponential form for 327.462.

$$3 \times 100 + 2 \times 10 + 7 \times 1 + 4 \times \frac{1}{10} + 6 \times \frac{1}{100} + 2 \times \frac{1}{1000}$$

$$3 \times 10^2 + 2 \times 10^1 + 7 \times 10^0 + 4 \times 10^{-1} + 6 \times 10^{-2} + 2 \times 10^{-3}$$

2. Review fraction-decimal conversion, and combine with mixed number-decimal conversion in pairs.

Example:

$$\begin{cases} \frac{3}{8} = 8\overline{)3.00} = .375 \\ 2\frac{3}{8} = 2 + \frac{3}{8} = 2 + .375 = 2.375 \end{cases}$$

As an alternative to conversion by division, a table of fraction-decimal equivalences may be provided for use in this area. Such a table is provided below.

Example:

$\frac{1}{2} = .5$	$\frac{1}{6} = .16\frac{2}{3}$	$\frac{1}{10} = .10$
$\frac{1}{3} = .33\frac{1}{3}$	$\frac{1}{7} = .14\frac{2}{7}$	$\frac{1}{11} = .09\frac{1}{11}$
$\frac{1}{4} = .25$	$\frac{1}{8} = .125$	$\frac{1}{12} = .08\frac{1}{3}$
$\frac{1}{5} = .20$	$\frac{1}{9} = .11\frac{1}{9}$	

and convert as follows:

$$\frac{3}{8} = 3 \times \frac{1}{8} = 3 \times .125 = .375$$

3. Emphasis should be placed on the fact that every rational number has a decimal equivalence that has a pattern that repeats without end.

Example: $\frac{1}{2} = .500\overline{0} \ldots\ldots\ldots\ldots$

$\frac{1}{3} = .3\overline{33} \ldots\ldots\ldots\ldots$

$\frac{1}{7} = .\overline{142857142857} \ldots\ldots$

$6 = 6.0\overline{0} \ldots\ldots\ldots\ldots$

4. Using the previous concepts, students should develop a fairly good proficiency in performing the four operations on decimal numbers. Problems in addition and subtraction of decimals should be presented to the students in linear form, requiring them to do the column arrangement in order to emphasize the neat and orderly arrangement of the decimal points in these two operations. For multiplication and division, the following approach might be helpful for developing within the student an understanding of the procedure for handling the decimal point.

Example:

$$\begin{cases} \frac{1}{10} \times \frac{1}{10} = \frac{1}{100} \\ .1 \times .1 = .01 \end{cases}$$

$$\begin{cases} \frac{1}{100} \times \frac{1}{10} = \frac{1}{1000} \\ .01 \times .1 = .001 \end{cases}$$

$$\begin{cases} \frac{1}{100} \times \frac{1}{100} = \frac{1}{10000} \\ .01 \times .01 = .0001 \end{cases}$$

Example:

$$\begin{cases} \frac{1}{10000} \div \frac{1}{100} = \frac{1}{100} \\ .0001 \div .01 = .01 \end{cases}$$

$$\begin{cases} \frac{1}{1000} \div \frac{1}{100} = \frac{1}{10} \\ .001 \div .01 = .1 \end{cases}$$

$$\begin{cases} \frac{1}{100} \div \frac{1}{10} = \frac{1}{10} \\ .01 \div .1 = .1 \end{cases}$$

In order to validate the mechanical process of moving the decimal point the same number of places in the divisor and the dividend, it might be well to approach the concept slowly, as follows (emphasize multiplication by the identity):

$\frac{14}{8} = \frac{140}{80} = \frac{1400}{800}$, etc. equivalent fraction concept

$\frac{1.6}{.8} = \frac{16}{8} = \frac{160}{80}$, etc. equivalent fraction concept

$\frac{47.28}{3.9} = \frac{472.8}{39}$ equivalent fraction concept

Then show the equivalent of : $\frac{47.28}{3.9} \times \frac{10}{10} = \frac{472.8}{39} = 39\overline{)472.8}$

J. Problem solving

1. Teachers are encouraged to review those skills and concepts related to problem solving that were developed in the section on integers. Any reteaching that seems necessary should be done before proceeding to more challenging problems.

2. Students should develop adequate proficiency in finding solution sets of open sentences containing noninteger rationals similar to the following:

$$x + \frac{1}{2} = \frac{3}{4}$$

$$\frac{3}{4}x + 5 = 7$$

$$\frac{2}{3}x + \frac{3}{2} = \frac{5}{4}$$

$$x + .28 = .78$$

3. Problems selected should have easily understood language, uncomplicated structure, and rational numbers that can be manipulated with ease.

UNIT V. Metric Geometry

A. Use a protractor in angle measurement - Teachers cannot assume that students have the ability to use the protractor. Careful instructions in its use should be emphasized since students tend to use the protractor with a limited degree of accuracy. Therefore, utilize activities aimed at increasing the skill of using the protractor, such as finding the sum of the angles in a triangle.

B. Constructions - The following constructions are suggested: duplicating a line segment, duplicating an angle, bisecting a segment, bisecting an angle, and constructing a perpendicular to a line from a point on the line and from a point outside the line.

For additional related activities, reference may be made to the Bureau of Secondary Curriculum Development's handbook on *Measurement*.

C. Perimeter - Teachers may wish to approach the perimeter concept by involving the student in the task of finding perimeters through actual measurement of the sides of various plane geometric figures. The perimeter concept

should be formalized through the gradual development of applicable formulas for the perimeters of a:

1. square
2. rectangle
3. triangle

Since many students at this point will be overwhelmed by the many formulas encountered in the development of the perimeter concept, the following procedure is suggested:

a. The student should be provided with a list of perimeter formulas which is consistent with the type of classwork or homework problems he will encounter.

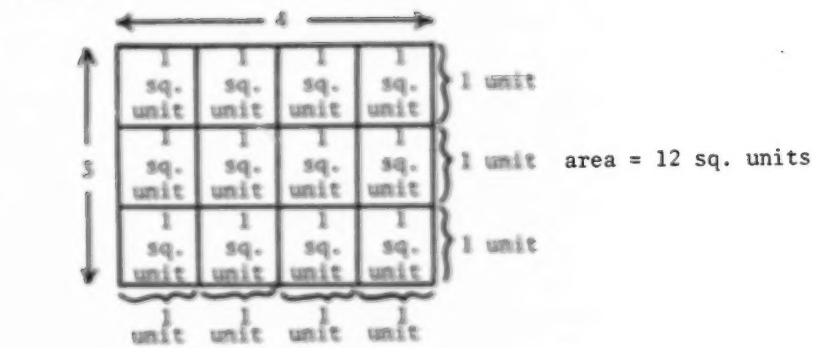
b. A concise methodology should be used by both teacher and student whenever a perimeter formula is being used. For example: perimeter of a rectangle = $2l + 2w$

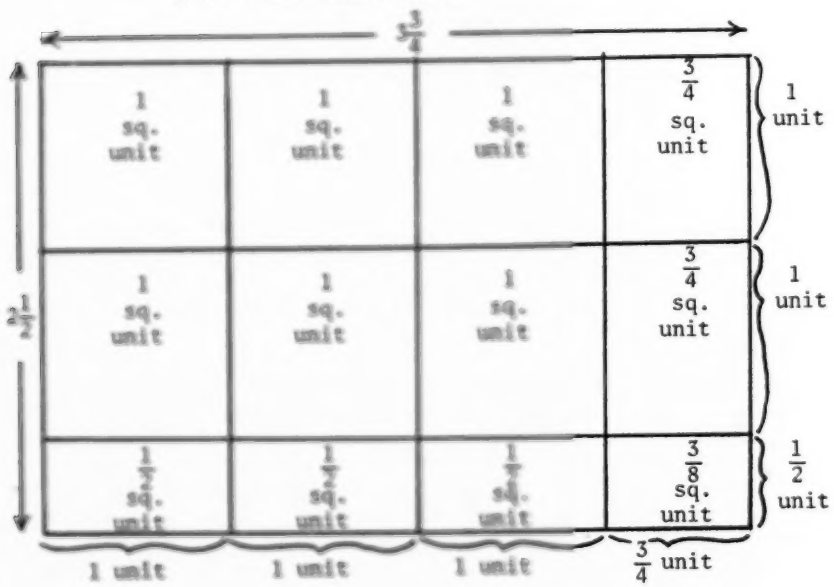
$$P = (2 \times 10) + (2 \times 4)$$
$$P = 20 + 8$$
$$P = 28$$

c. A properly labeled diagram should be used that represents the geometric figure being considered and which can be used to provide a check of the result found by using the perimeter formula.

D. Area - Discuss unit area and the fact that a side of a unit square may vary in dimension. At this point, it is possible to consider squares and rectangles whose dimensions are represented by whole numbers. The areas of these geometric figures may then be computed by adding up the number of unit squares it is possible to form. This concept can be extended to find areas of squares and rectangles whose sides are either represented by mixed numbers or mixed decimals.

Examples:





$$\text{area} = 6 \cdot 2\left(\tfrac{3}{4}\right) \cdot 3\left(\tfrac{1}{2}\right) \cdot \tfrac{3}{8}$$

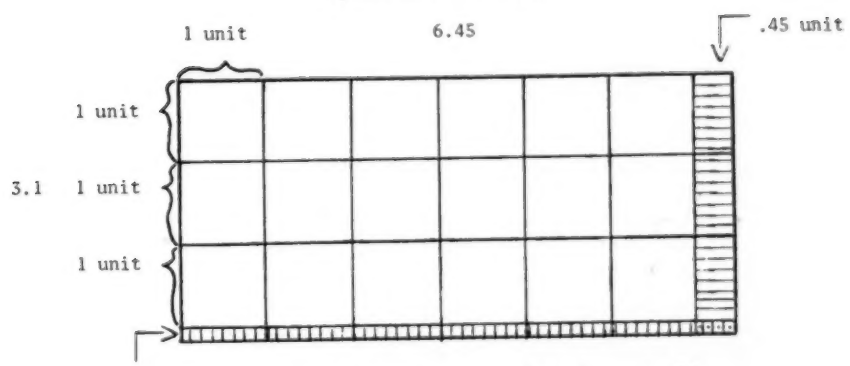
$$= 6 \cdot \left[\tfrac{6}{4} \cdot \tfrac{3}{2} \cdot \tfrac{3}{8}\right]$$

$$= 6 \cdot \left[\tfrac{12}{8} \cdot \tfrac{12}{8} \cdot \tfrac{3}{8}\right]$$

$$= 6 + \left[\frac{27}{8}\right]$$

$$= 6 + \left[3\frac{3}{8}\right] = 9\frac{3}{8} \text{ sq. units}$$

square units of area



18 unit squares	18 x 1	= 18.
3 rect. (.45 x 1)	3 x .45	= 1.35
6 rect. (.1 x 1)	6 x .1	= .6
1 rect. (.45 x .1)	.45 x .1	= .045
		19.995

square units in area

The area concept should now be formalized through the gradual development of applicable formulas for the areas of a:

1. square
2. rectangle
3. parallelogram
4. triangle
5. trapezoid

It is suggested that the students be provided with a list of these formulas and that the methodology of their usage, mentioned during the discussion of perimeters, again be carried out. Associated diagrams may now be used to

represent pertinent information. Some excellent commercially produced items that are helpful in the visualization of area concepts are available at nominal cost.

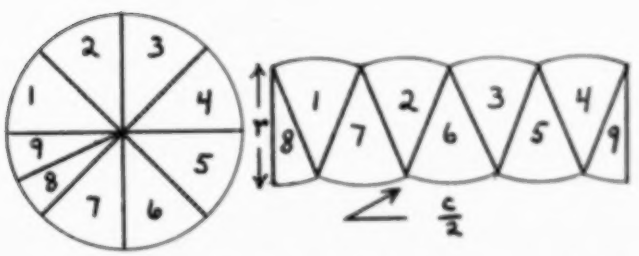
E. Circle

1. Discuss the circle and its parts.

2. To develop the concept that $\pi = c/d$, student involvement in the activity of measuring the circumference and diameter of a few circles and then finding the quotients c/d should be helpful in the realization that π is a number slightly larger than 3. Discuss the fact that accuracy in measurement and accuracy in computation will lead to the more accurate value of $\pi = 3.14$ or $\pi = 3.1416$, although an ultimate in accuracy in this computation will never be reached.

   A gradual and careful transition will need to be made from $c/d = \pi$, to $c = \pi d$, and hence to $c = 2\pi r$.

3. To develop the formula for area of a circle, the teacher may approach it through the rectilinear rearrangement of n equal sectors of a circle, of which one is bisected, and making use of the formula for the area of a rectangle.



$A = l \times w$

$A = \frac{c}{2} \times r$

$c = 2\pi r$

$A = \frac{2\pi r}{2} \times r$

$A = \pi r^2$

NOTE: See Bureau of Secondary Curriculum Development's handbook on *Measurement*.

Class discussion should now concentrate on what happens to the rectilinear form as n becomes very large.

F. Denominate numbers - There should be some review of operations with denominate numbers with particular application to perimeter and area. Students should become aware that addition and subtraction of denominate numbers may be accomplished with little difficulty, and that multiplication and division of denominate numbers will necessitate changing them into numbers having a single unit of measure.

G. The Pythagorean theorem - In selecting right triangles for illustrating the Pythagorean theorem, care should be taken that these triangles have sides whose measurements form Pythagorean triplets. Some examples are:

3,4,5
6,8,10
9,12,15
5,12,13
8,15,17

UNIT VI. Ratio, Proportion, and Percent

A. Ratio - A ratio is the comparison of two numbers by division, provided the two numbers have the same units. If the numbers have different units, then it is called a rate. Students should learn the various ways to express a ratio, i.e., 4 is to 6; 4:6; 4/6. Students should be encouraged to look at a ratio for the possibility of a quick reduction by inspection. They should not, however, attempt to reduce a ratio whose reduction is not easily apparent.

B. Proportion - A proportion is defined as the equality of two ratios. Students should become familiar with the various ways of expressing a proportion, i.e., 6 is to 4 as 15 is to 10; 6:4 = 15:10; 6/4 = 15/10. The student should learn how to distinguish the first, second, third, and fourth terms of a proportion. He should clearly understand the terms *means* and *extremes*. It is important to point out that if the first and third terms of a proportion are equal, then the second and fourth terms are equal, and conversely. When one of the terms of a proportion is a variable, then the proportion is a special case of open sentence. As a special case of open sentence, there is a special technique of solution called

the cross product rule. Students should be encouraged to examine carefully the ratios of a proportion for possible reduction before applying the cross product rule.

Example: $21/49 = x/14$ $\quad 3/7 = x/14$
$49x = 294$ $\quad 7x = 42$
$x = 6$ $\quad x = 6$

In proportions of the following type, solutions can be found by inspection.

Example: $17/183 = x/183 \quad \therefore \quad x = 17$

$19/41 = 19/x \quad \therefore \quad x = 41$

The cross product rule can also be used to determine ordering of rational numbers.

Example: $3/5 \; ? \; 5/8$ $\qquad -2/5 \; ? \; -6/7$
$3 \cdot 8 \; ? \; 5 \cdot 5$ $\qquad (-2)(7) \; ? \; 5(-6)$
$24 < 25$ $\qquad -14 \; ? \; -30$
$-14 > -30$

$\therefore \; 3/5 < 5/8$ $\qquad \therefore -2/5 > -6/7$

C. Percent - Review with students the fact that percent means hundredths, and the equivalence relationship, i.e., $80\% = 80/100 = .80$. Show the use of a proportion to solve problems involving percent.

Example: What number is 60% of 40?

$x/40 = 60/100$
$x/40 = 6/10$
$10x = 240$
$x = 24$

In the above example, point out the advantage of reducing 60/100, the use of the cross product rule, and the final solution by applying the axiom of division. Emphasize that the cross product rule and the axiom of division are adequate for the solution of all proportions. It can also be shown that percent problems can be solved by direct translation into a mathematical sentence.

Example: 60% of what number is 36?

$.60x = 36$

D. Related activities - Many problems related to the following areas lend themselves well to solution by means of proportion:

Lever	Similar triangles
Density	Scale drawings
Force	Machine shop problems
	Circle graphs

UNIT VII. Probability and Statistics

A. Probability - Students should develop a clear understanding of what probability is. This concept is most commonly approached through coin-tossing experiments, drawing a particular card from a deck of 52, use of a spinner, and other similar activities. These activities should lead to the realization that probability is the number of ways in which a particular event can happen divided by the total number of possibilities. The difference between mathematical (a priori) and empirical (a posteriori) probability should be made clear. Interest can be heightened by having student participation in activities that will reinforce the empirical probability concept. A few such activities follow:

Activity 1:

A box contains 5 discs (use checkers or poker chips). Three of them are numbered "1" and two of them are numbered "5." A second box contains five discs, two are numbered "10," one numbered "1," and two numbered "5." A student picks a disc from each box.

What is the probability of getting a sum of 10 from the 2 discs?

The teacher should make it quite clear what the activity is. The students should be asked to guess the probability before conducting the experiment. Let the student pick these discs about 20-30 times, and record the results.

The mathematical (a priori) probability for this activity is illustrated by the following table:

	1	1	1	5	5
1	2	2	2	6	6
5	6	6	6	10	10
5	6	6	6	10	10
10	11	11	11	15	15
10	11	11	11	15	15

(2nd box of discs)

(1st box of discs)

$4/25 = 16^{\%}$

It is clear that there are 25 outcomes of this experiment and four of these will give a sum of 10. If the empirical result is not "close" to the mathematical result, this could lead to a discussion of "why not."

Many variations of this activity can be performed.

Activity 2:

A thumbtack, roofing nail, or cone-shaped paper cup can be used for this activity. When tossed, the tack can either land point up or point down. Through the empirical approach, have three students determine the probability of its landing with the point down.

Recording and interpretation of the data are important. In this activity the student cannot compute the mathematical (a priori) probability because he does not know enough physics.

Activity 3:

If three beads are thrown from a cup 20 or 30 times, what is the empirical probability of them being the vertices of an obtuse triangle? (All collinear arrangements must be counted as no throw.)

Activity 4:

On a sheet of paper, mark off a series of lines 2 inches apart; taking 25 sticks 1-inch long, hold them about 15 inches above the paper, and allow them to fall. Count the number of sticks that cross or touch a line. Repeat 20 or 30 times. The number of sticks crossing or touching a line divided by the total number of sticks dropped

is an interesting probability since its reciprocal is a number that approaches the value for π.

Activity 5:

Have each student in class obtain a frequency distribution of the last digit for 25 telephone numbers (different numbers for each student). Have each student find the average value of the last digit and compare it with 4.5, the theoretical value if all digits are equally likely. A class average might then be computed and compared to 4.5. This experiment also lends itself very nicely to a study of measures of central tendency.

Mathematical probability should be restricted to simple problems involving coins, cards, spinners, marbles, dice, etc.

8. Statistical tables - The primary object of a table is to present large amounts of data in a precise and orderly manner. An effective table, therefore, is one which contains a systematic listing of relevant information in a minimum amount of space. Statistical data lend themselves well to organization and presentation by means of frequency tables and frequency histograms. Statistical data also are well suited to a study of measures of central tendency such as the mean, mode, and median. Students should acquire the ability of accurately determining the measures of central tendency for data that are reasonably uncomplicated. They should become aware that the measures of central tendency are not always truly descriptive. For example, given the grades 50,60,60,65,95,97, and 98, the mean is 75, the mode is 60, and the median is 65, none of which provides measures that are descriptive of the data. Students should also be aware that the histogram provides an immediate visual interpretation of central tendency or lack of it.

UNIT VIII. Informal Geometry

A. Point, line, plane - The students should be made aware that point, line, and plane are undefined, abstract, geometric concepts. They are commonly associated with some analogous real life situations with which students are familiar. For example, a point may be made analogous with a position or location in space, a line is analogous with a stretched string, and a plane analogous to a sheet of paper or other flat surface. Students should learn that there exists an infinity of points on a line. This concept can be reinforced by relating the points on a line to the number line and showing that for any two rational numbers, there always exists a third between them. Hence, there is an infinity of numbers between any two points on a line. Students should know that two distinct points will determine one and only one straight line. Some discussion should be given to the positioning of a line with respect to a particular frame of reference, i.e., horizontal, vertical, diagonal, or oblique. The concept of intersecting and parallel lines is probably well established and should need little review. The tetrahedron is a fine model to demonstrate some important relationships that exist among points, lines, and planes.

   For example, any three of the four vertices are noncollinear points that determine unique planes, the edges are lines that represent the intersection of pairs of planes, and the vertices are points that represent the point of intersection of three planes. The tetrahedron encloses a portion of space isolated from the whole of space by four intersecting planes.

B. Triangles and polygons - In discussing the triangle, the teacher should emphasize the concept that three noncollinear points determine a unique triangle. The meaning of non-collinear as it applies here will need to be clarified. The student should learn that three noncollinear points mean that all three points do not lie in the same straight line since each pair of points determines a straight line. This concept will be important for the later development of the idea that given n noncollinear points, for which the same restriction holds, they will determine an n-sided polygon. Students should learn that triangles may be named relative to their angles or relative to their sides, i.e., a right triangle contains one and only one right angle, an obtuse triangle

contains one and only one obtuse angle, and an acute triangle contains three acute angles. In a similar manner, an equilateral triangle has three sides that are equal, an isosceles triangle has two sides that are equal, and a scalene triangle has three sides that are unequal. There should be some discussion of the various ways in which triangles interrelate. It might be wise to provide students with the following chart to aid them in understanding these interrelationships:

A scalene triangle can be { acute, obtuse, right }

An isosceles triangle can be { acute, obtuse, right }

An equilateral triangle must be acute.

An acute triangle can be { scalene, isosceles, equilateral }

An obtuse triangle can be { scalene, isosceles }

A right triangle can be { scalene, isosceles }

In a general discussion of polygons, the following points should be clearly emphasized:

1. Given n distinct points, for which it is true that collinearity exists for pairs of points only, they will determine a polygon of n sides.

2. Any polygon that does not contain at least one angle that is greater than 180° and less than 360° is convex.

3. Any polygon that does contain such a reflex angle is concave.

4. A polygon divides a plane into two areas, one interior to the polygon and the other exterior to the polygon.

Students should become familiar with the names of the more common quadrilaterals, as well as the pentagon, hexagon, and octagon. They should learn the meaning of regular polygon and be able to draw regular polygons of n sides by selecting points on the circumference of a circle so as to divide the circle into n equal arcs and then connecting the points in succession.

C. Figures of three dimensions - In discussing concepts in three dimensions, be sure that students understand the following relationships:

1. A line intersecting a plane
2. A line that lies in a plane
3. A line that is exterior to a plane, i.e., a line that is parallel to a plane
4. Intersecting planes

Students should become familiar with the rectangular solid and the proper naming of its vertices, edges, and faces as well as the number of each. There should be a discussion of the cylinder, cone, pyramid, and sphere as fairly common examples of solids that enclose a portion of space.

UNIT IX. Use of the Slide Rule

Although slide rules vary in their complexity and cost, a very inexpensive type generally exhibits an A, C, and D scale which is all that is necessary to perform the operations of multiplication, division, the squaring of a number, and finding the square root of a number.

A. Examination of C and D scales - By removing the center piece from the slide rule, an examination of the D scale reveals that the first 10 counting numbers are represented from left to right along the scale. The numbers representing the

largest subdivisions between the numbers 1 and 2 are printed smaller than the rest of the numbers on the scale and are not to be considered when pointing out the first 10 counting numbers. When the number 10 is reached, it is noted that it is represented on the slide rule as a 1 and indicates that to continue this counting procedure one must return to the initial starting point at the left-hand side of the scale. At this point, the interpreting of the subdivision between numbers is necessary. Continuing to count, however, will reveal the nature of the various subdivisions between the first 10 counting numbers. This counting procedure should be extended to include the first 100 counting numbers.

It is suggested that a class activity now center around the location of larger counting numbers such as 250, 310, 870, etc. The location of decimal places on the slide rule should be discussed thoroughly at this point, as there are no decimal points on the slide rule. Thus, on the D scale,the location of .16, 1.6, 16, and 160 will be identical. The student should read this as "one, six" rather than "sixteen." When the center piece of the slide rule is reinserted, the students should be made aware of the fact that the C and D scales of the slide rule are identical.

B. Multiplication - The basic procedure for multiplication on the slide rule is as follows:

1. Locate the multiplier on the D scale.
2. Move the center piece so the number 1 on the C scale is directly over the multiplier.
3. Locate the multiplicand on the C scale.
4. Move the cursor so the hairline is directly over the multiplicand.
5. Read the product under the hairline on the D scale.

The direction in which the center piece is moved is determined by estimating whether the resulting product will be to the left or the right of the multiplier. However, it becomes quite evident in many problems that movement to the right with the center piece is incorrect, as the multiplicand on the C scale is entirely off the D scale. For example, when multiplying:

1. 2 x 4, move center piece to the right.
2. 2 x 6, move center piece to the left.
3. 2 x 5, direction of movement does not matter.

It is suggested that the level of difficulty of multiplication problems gradually be increased to include decimal numbers. At first, it is suggested that these problems involve no estimating on the part of the student in terms of the location of the multiplier, multiplicand, or final product. Estimation in the sense of a correct location of a decimal point, however, is a necessary operation and should be stressed.

When a problem such as the multiplication of 2.25 x 4 is introduced, the student should be made aware of the fact that not all numbers can be represented exactly by a mark or line on the position of the scale with which he is working, and that many times an estimation is called for. A reexamination of the subdivision of the scale being used will quickly help to verify the value of each interval. In this case, the first task would be to locate 2.20 and 2.30 and then examine the number of intervals between these two numbers to determine their value. As there are five intervals, each interval represents .02. Therefore, the student should now locate 2.24 and 2.26. His estimation as to the location of the multiplier can now take place with some degree of accuracy. It is suggested that the procedure involved in estimating a multiplier, multiplicand, or product be demonstrated by the teacher in several examples and that gradually the level of difficulty of these problems be increased. In the multiplication of a problem such as 1.4 x 4.2, it may be pointed out to the student that multiplying the last two digits together mentally will assist in the estimation of the final digit of his answer. The first two digits of his answer can be read directly from the slide rule as 58 and the third digit provided by estimation, verified by this mental arithmetic.

C. Division - The basic procedure for division is as follows:

1. Locate the dividend on the D scale.
2. Place the divisor on the C scale directly over the dividend.
3. Move the cursor so the hairline is over the 1 on the C scale.
4. Read the quotient under the hairline on the D scale.

Class discussions concerning decimal point location and a gradual increase in the level of difficulty of assigned problems are again suggested.

D. Squaring a number - The operation of squaring a number should be discussed in terms of the meaning of the number 2 when used as an exponent.

Examples: $6^2 = 6 \times 6 = 36$

$(1.2)^2 = 1.2 \times 1.2 = 1.44$

The A scale is commonly used with the D scale to perform squaring and square root problems and, therefore, the center piece may be removed to avoid confusion with other scales. The A scale should be examined in terms of its subdivisions, and the fact that the left- and right-hand portions of the A scale are identical. It should be noted that numbers whose square is less than 10 will appear on the left portion of the A scale, while numbers whose square is greater than 10 but less than 100 will appear on the right-hand portion of the A scale. When finding the square of a number, it is suggested that to help find the proper location of the decimal point that an approximation technique be used. Thus, when squaring 68.6, the squares of 60 or 70 should be considered to help to determine the relative size of the outcome. The procedure involved in the squaring of a number is to locate the number on the D scale, place the hairline over this number, and read the square of the number on the A scale. After a sequence of problems is worked out, the students may check their answers by inserting the center piece and performing the multiplications on the C and D scales.

E. Finding square roots - In finding the square roots of numbers, the students should be instructed to use exactly the opposite scales compared to squaring numbers in location of number and final result. It may be noted here that the square roots of all numbers will be read on the D scale. However, in locating numbers of which one wishes to take the square root, one must be conscious of the two identical portions of the A scale. This problem is easily overcome by the fact that all numbers containing an odd number of digits (the number of digits considered are only those to the left of the decimal point) will be located on the left-hand portion of the A scale, while all numbers containing an even number of digits will be located on the right-hand portion of the A scale.

UNIT X. Recreational Mathematics

A study of mathematics is full of interesting, amusing, entertaining, and sometimes perplexing ideas which lend themselves to what is commonly referred to as *recreational mathematics*. The introduction of these recreational topics at frequent intervals during any course in mathematics is a worthwhile endeavor. If chosen wisely, they can be interesting and enjoyable for the student and they, quite often, reawaken interest and create positive motivation toward the study of mathematics. Quite frequently, they result in the student's acquisition or mastery of valuable mathematics concepts. The following are a few of these recreational ideas. Many more can be found in the ample literature that is available in this area.

A. Prime numbers - These offer a fruitful area for some interesting examples. Consider the primes, 7, 17, 29. Their reciprocals are rational numbers whose repeating decimal expansions contain 6, 16, and 28 digits, respectively. This is a good illustration that division by n can produce at most (n - 1) different remainders. The decimal expansion of 1/7 is .142857142857 ..... If the digits of the repeating pattern are multiplied by 1, 2, 3, 4, 5, 6 in succession, a cyclic permutation of these digits is produced.

Example:
1 x 142857 = 142857
2 x 142857 = 285714
3 x 142857 = 428571
4 x 142857 = 571428
5 x 142857 = 714285
6 x 142857 = 857142

The same thing is true for 1/17 and 1/29.

The fraction 1/17 has the repeating pattern 5882352941176470, and multiplication by any whole number 1 through 16 will produce cyclic permutation.

The fraction 1/29 has the repeating pattern 344827586206896-551724137931O, and multiplication by any whole number 1 through 28 will produce cyclic permutation.

B. Magic squares - These have fairly widespread appeal. The construction of an n x n magic square, where n is any odd

whole number equal to or greater than 3, is a fairly easy process. Using the first n^2 natural numbers, the number 1 is placed in the center square of the top row and succeeding natural numbers are placed in squares found by going up 1 row and to the right 1 column. In order to accomplish this, one must think of the rows and columns as though they existed on cylinders. If one moves up one row from the top row, he will then arrive in the bottom row. If one moves one column to the right of the extreme right column, he will arrive in the extreme left column. If by this process, one arrives in a square that is already occupied by a number, he must return to the square just left and go directly underneath.

Example:

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

An infinite number of magic squares can be produced from the one above by adding the same integer to each of its elements or by multiplying or dividing each of its elements by the same integer excluding zero.

A different type of magic square can be formed that has the unique property of forcing a predetermined number.

Example: Consider the number 24 and this matrix:

11	8	12
9	6	10
6	3	7

Choose any number, say 10, cross out the remaining numbers in that row and column, i.e., 12, 7, 6, and 9. This leaves 11, 8, 6, and 3. Choose a second number, say 8, and again cross out the remaining numbers in that row and column which means 11 and 13. Now only 6 is left. If this last remaining

number is added to the previously chosen numbers, the sum is 24, i.e., 6 + 8 + 10 = 24. This will always happen, regardless of the three numbers chosen or the order in which chosen. This matrix was constructed from the following addition table:

+	4	1	5
7			
5			
2			

Squares such as these can be constructed as 3 x 3, 4 x 4, 5 x 5, or any n x n. The only requirement is that 2n numbers be chosen such that their sum equals the number to be found. They can also be constructed using multiplication instead of addition. Positive and negative integers and fractions can also be used.

C. Paper cutting tricks - These tricks have interesting and sometimes startling results that give insight into some elementary topological concepts.

Example: Consider the question, "Can a hole be cut in a piece of paper such that the hole is larger than the paper itself?" The answer, of course, is "yes." As a matter of fact, an 8½ x 11-inch piece of paper can easily be cut so as to produce a hole that is nearly 10 feet in diameter. To accomplish this, one must cut an 8½ x 11-inch sheet so as to form a continuous strip ½-inch wide. Then, starting ¼-inch from either end, cut a slit down the middle of this ½-inch strip stopping ¼-inch from the other end. The result will be a continuous ¼-inch strip that is approximately 30 feet long. This strip can form a circle with a hole that is nearly 10 feet in diameter.

D. Palindrome numbers - These are numbers that read the same both ways. Ex. 22, 212, 142241. Start with any number, and add the number obtained by reversing the digits. Continue as often as necessary and eventually a palindrome number will result.

Example:

32	154	253
23	451	352
55	605	605
	506	706
	1111	1311
		5151
		6666

This activity can result in students trying to find a number which takes the largest amount of addition steps to result in a palindrome. It gives them practice in adding.

Although this is not a palindrome number, it is an activity which gives the student practice in subtraction.

Example: Pick any four digit number, such as 1294. Arrange it from highest to lowest digits, i.e., 9421; then arrange it from lowest to highest digits, i.e., 1249, and subtract the smaller from the larger.

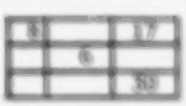
$$\begin{array}{r} 9421 \\ 1249 \\ \hline 8172 \end{array}$$

Continue this process as necessary until an interesting result is obtained. What is it?

8721	7443	9965	6642	7641		7641
1278	3447	3699	2466	1467		1467
7443	3996	6264	4176	6174	-	6174

E. Cross number puzzles - These are similar to crossword puzzles except that digits are used to form numbers, rather than letters to form words. Many types of cross number puzzles can be given to help students to practice in many topics, i.e., four operations, percent, volume. There are many references available on this material.

F. Number puzzles - This is essentially finding the missing addend. It is very appropriate for practice in addition-subtraction facts and solving simple equations intuitively. The teacher prepares the following:



The student must fill in the five blank cells so that the rows and columns form addition sentences.

8	9	17
7	6	13
15	15	30

After operations with signed numbers have been developed, this exercise can be extended to give practice in that area.

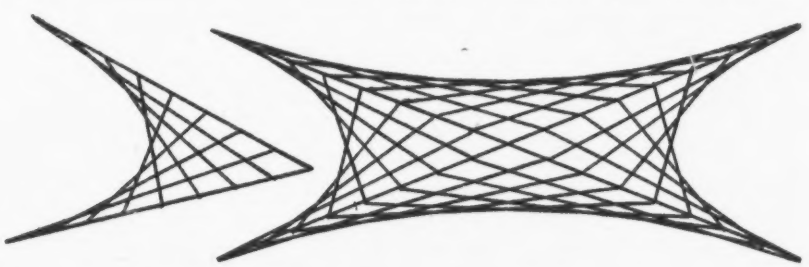
7		3
19		10

G. Coordinate pictures - One fun activity is to make coordinate pictures as suggested in Unit I. Then, multiply each first member of the ordered pairs by 2, and then plot the new set of points. What happens to the picture? Another variation is to ditto a nonrectangular grid system and plot the points of a picture. How does this affect the picture?

H. Building numbers - This is like a game of solitaire. The rules are: You must use four numeral 4's, and you may use a positional notation, operational symbols, and parentheses to "build" numbers. Can you "build" the integers 1-25?

$$\frac{44}{44}, \quad \frac{4}{4}+\frac{4}{4}, \quad \frac{4+4+4}{4}$$

I. Curve stitching - On a piece of oak tag, draw an angle. Now, mark off equal segments on each ray. Using a sharp object, punch holes at each mark. Thread string or wool from the farthest hole on one ray to the nearest hole of the other. Keep repeating this process. When you are finished, the result looks like the first figure. Can you make the second?



This may be extended into three-dimensional shapes by using cardboard boxes.

J. Cutting a cube - A 3-inch cube is painted blue. Answer the following questions.

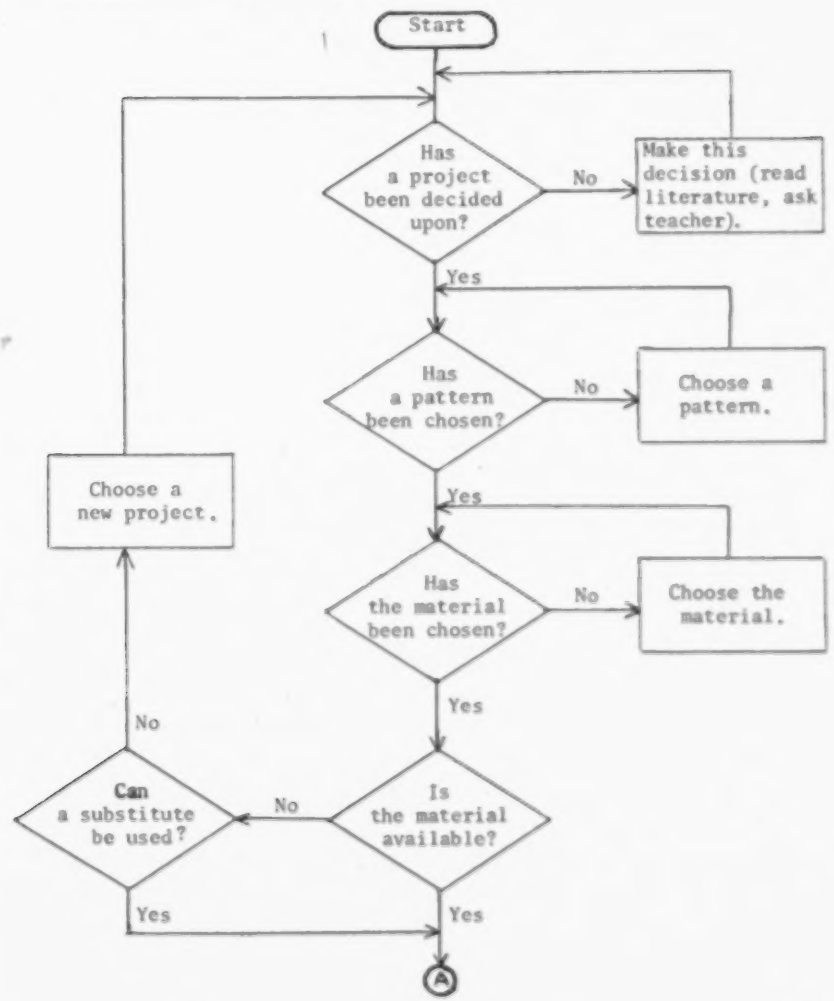
1. How many cuts are necessary to divide the cube into 1-inch cubes?
2. How many cubes would you have?
3. How many cubes would have *four* blue sides?
4. How many cubes would have *three* blue sides?
5. How many would have *two* blue sides?
6. How many would have *one* blue side?
7. How many cubes would have *no* blue sides?

UNIT XI. Flow Charts and Calculators

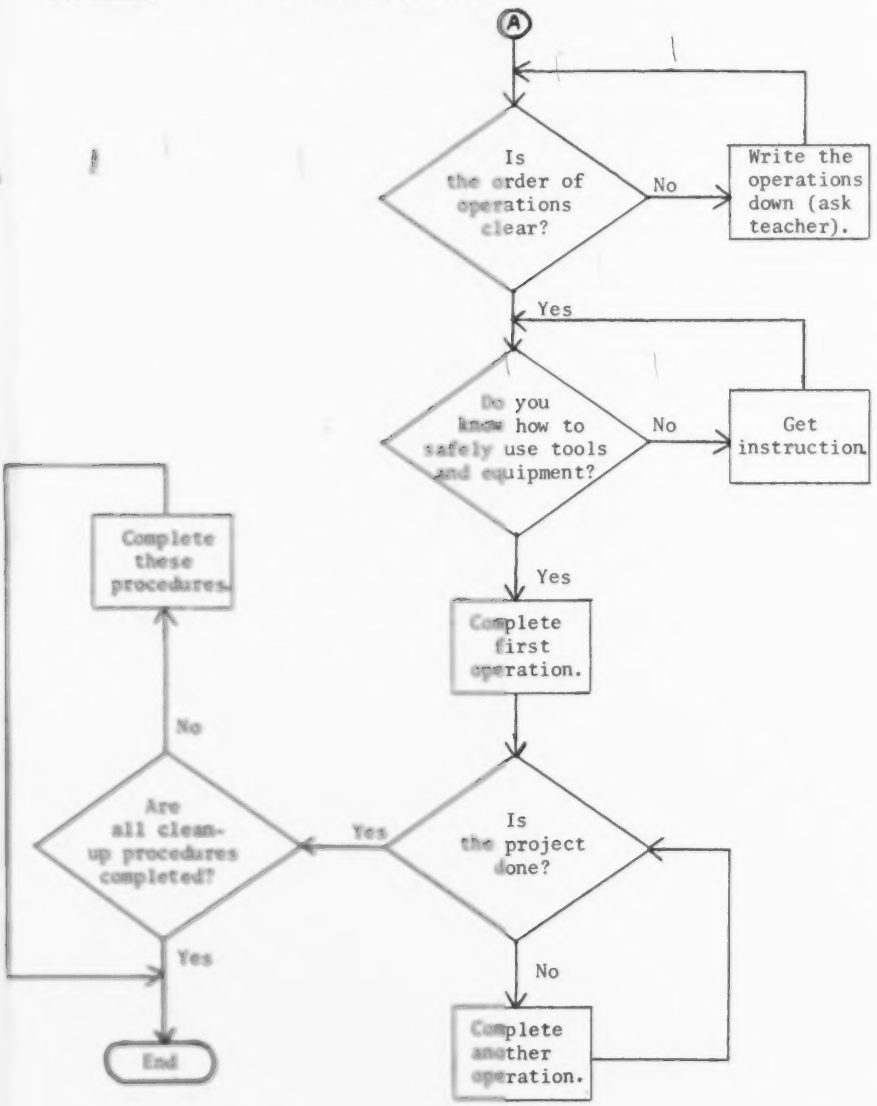
A. General discussion - Flow charts are most frequently introduced as a device for developing the elementary skills of programing. However, flow charts may be used in reference to many areas of problem solving which are both mathematical and nonmathematical in nature. The development of a flow chart necessarily entails a thorough analysis of a problem or task and an understanding of related devices that might be used in arriving at a solution. To this extent, flow charting helps to develop methodology and analytical thinking on the part of those who engage in it. The operation of a desk calculator lends itself very nicely to the concept of flow charting. If desk calculators are available for student use, the student should be encouraged to formulate specific flow charts for specific problems, i.e., the specific flow chart will contain statements such as "enter 146.32" instead of "enter first addend." When students acquire the skill of formulating specific flow charts, they will be in a better position to appreciate the advantages of generalizing and should then be guided in that direction.

B. Examples of flow charts:

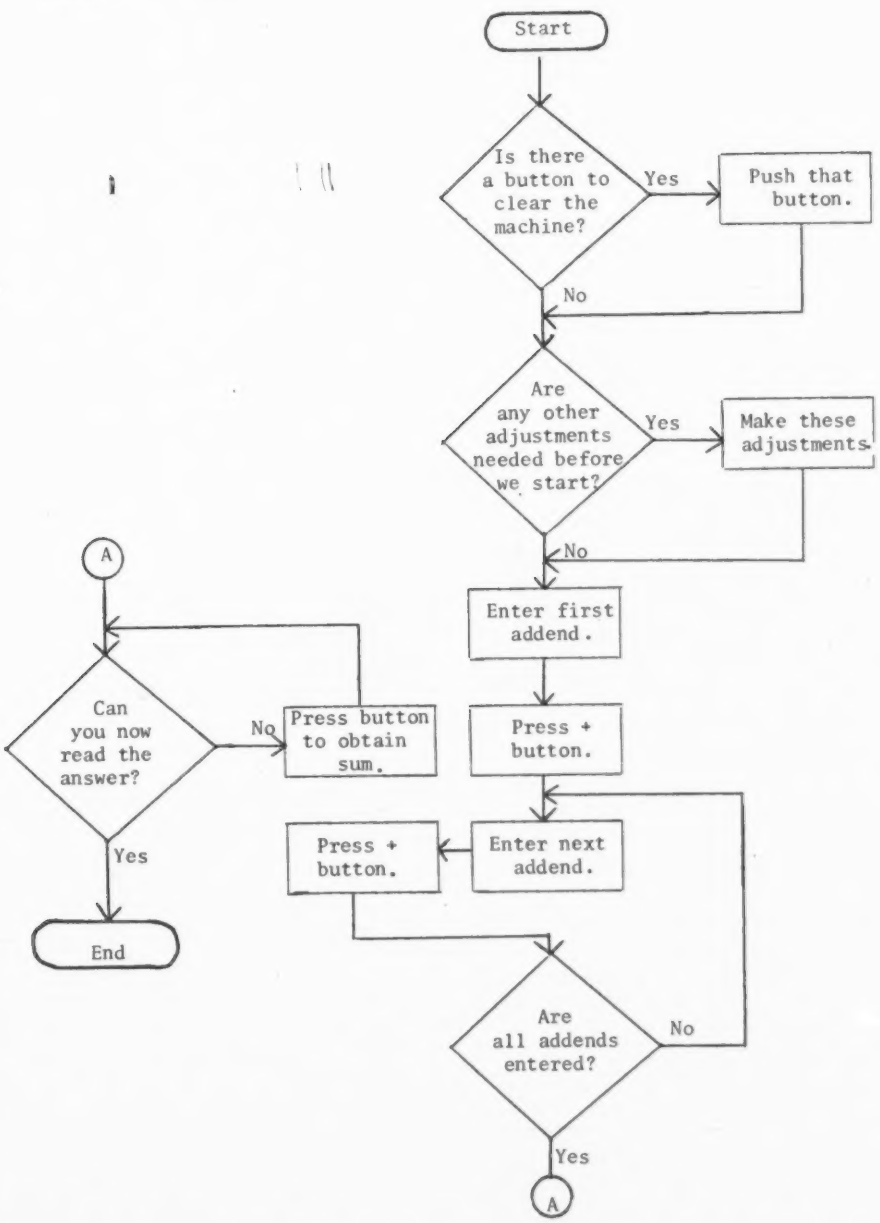
Example 1: Work on a shop project



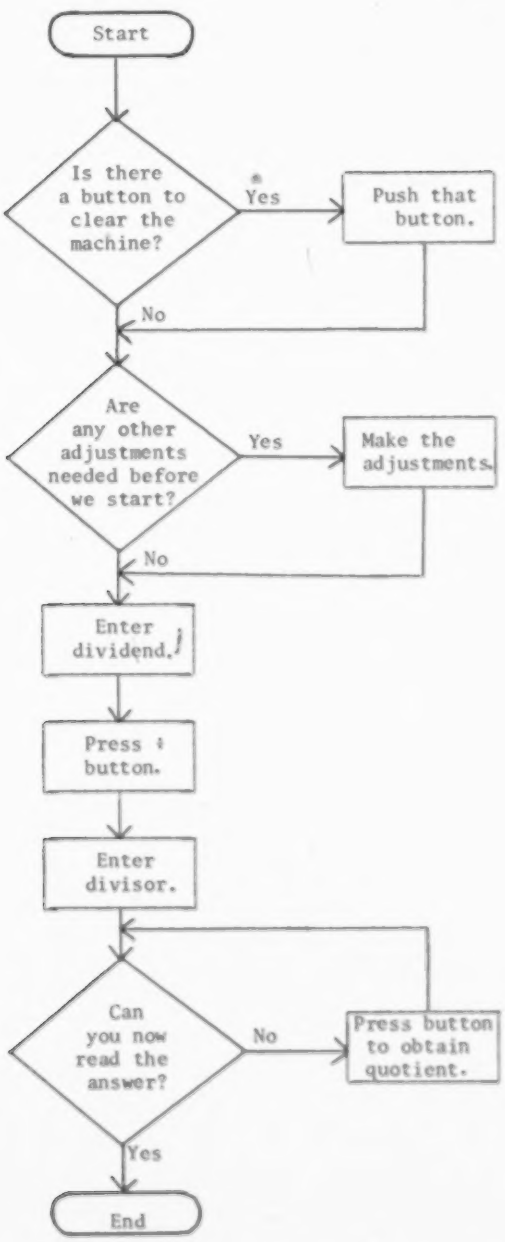
Example 1: Work on a shop project (con.)



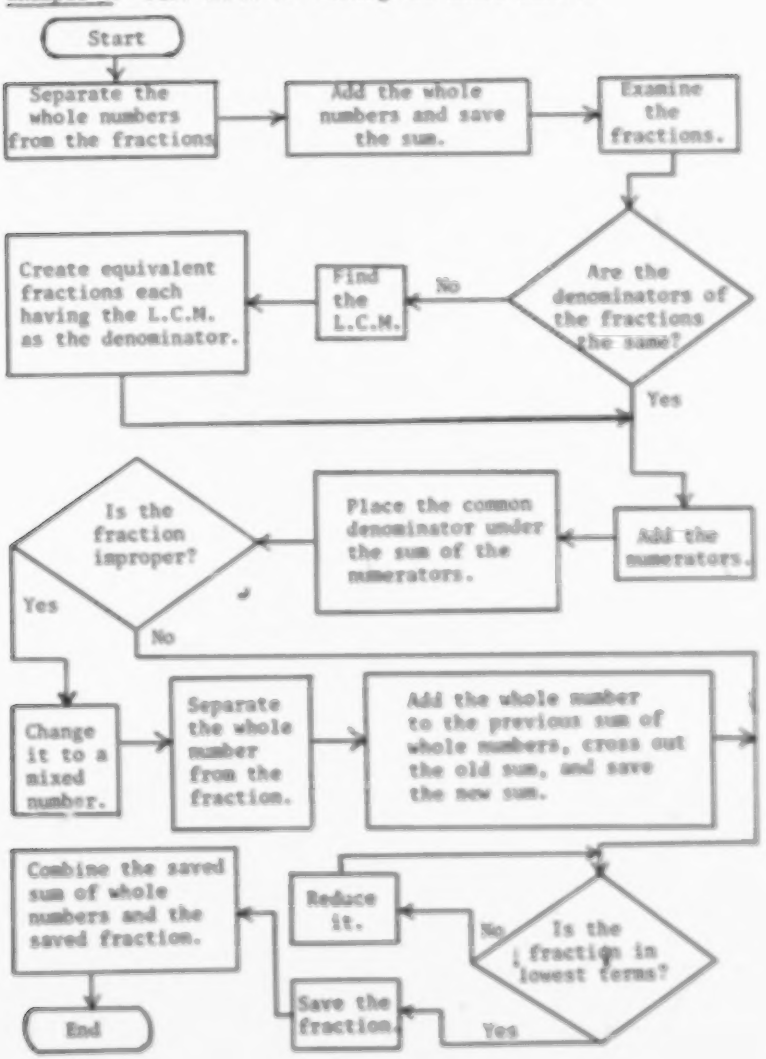
Example 2: Addition on a desk calculator



Example 3: Division on a desk calculator



Example 4: Flow chart for adding two mixed numbers


Start
Separate the whole numbers from the fractions
Add the whole numbers and save the sum.
Examine the fractions.
Are the denominators of the fractions the same?
No
Find the L.C.M.
Create equivalent fractions each having the L.C.M. as the denominator.
Yes
Add the numerators.
Place the common denominator under the sum of the numerators.
Is the fraction improper?
Yes
No
Change it to a mixed number.
Separate the whole number from the fraction.
Add the whole number to the previous sum of whole numbers, cross out the old sum, and save the new sum.
Is the fraction in lowest terms?
No
Reduce it.
Yes
Save the fraction.
Combine the saved sum of whole numbers and the saved fraction.
End

UNIT XII. Volumes of Geometric Solids

A. The concept of volume - Both the concept of volume and the technique of finding volumes of geometric solids often present difficulties to many students. These difficulties may, to some extent, be overcome by having the students construct unit cubes that are 1 inch on an edge. Each student may make five or six so that the class will have a large number with which to work. The pattern for construction of unit cubes should be supplied by the teacher. A class activity may now be directed toward the building of cubes and rectangular prisms whose volumes may be found by a direct counting of the number of cubic units needed to form them. Students' attention should then be directed to the fact that the volume in each case is equal to the number of cubic units in the bottom layer multiplied by the number of layers.

B. The general formula V = Bh - An easy transition can then be made to the generalized formula V = Bh, which may then be applied to the cube, rectangular prism, triangular prism, and the cylinder. To develop the concept that a cone has 1/3 the volume of a cylinder with equal base and height, and a pyramid has 1/3 the volume of either a cube or rectangular prism with equal base and height, appropriate models are most useful to demonstrate the relationship visually. Such models may also be used to validate the formula: $V = \frac{4}{3}\pi r^3$ for the volume of a sphere. In demonstrating this relationship, the classroom models necessary will be any sphere and the smallest cube in which that sphere can be enclosed. If water or some fine-grained material such as sugar or salt is used to fill the space around the sphere, it can be visually demonstrated that this material will fill approximately one half the cube after removing the sphere. If the formulas for the volume of a cube and sphere are used, and $\pi = 3.14$, then the calculated value of the space around the sphere will be very close to 48 percent of the volume of the cube. Much emphasis should be placed on expressing volume in cubic units; and students will need help in using the table of cubic measure in problems that may require conversions.

C. Suggestions to teachers - Throughout this unit on geometric solids, classroom models and related visual aids should be utilized extensively. This unit may well include student participation in the construction of the five regular

solids - tetrahedron, cube, octahedron, dodecahedron, and icosahedron. Students should be provided with a list of formulas for finding volumes of geometric solids, and emphasis should be placed on the proper use of the formulas. Simple numbers should be used so that the development of these concepts will not be too difficult.

UNIT XIII. Consumer Mathematics

A. Student interest - Consumer mathematics is a topic that can easily be adapted to many student interests and needs. Class discussions frequently lead into areas of consumer mathematics which prompt enthusiastic student participation.

B. Examples for discussion

1. A discussion that centers around automobiles can lead into many problem-solving areas such as purchase price, operating costs, and liability coverage, which is particularly relevant since teenagers must pay a very heavy premium. The purchase price is relative to make, model, dealer, and time of purchase and can lead to problems in installment buying, time payments, loans, interests, carrying charges, percent of savings by careful shopping, etc. Operating costs can lead to problems in budgeting, i.e., percent of operating cost that properly belongs in that part of the budget concerned with essential transportation and the part concerned with recreation. Liability coverage leads naturally into the general topic of insurance and related problem areas.

2. Dress and grooming is another area in which enthusiastic class discussion can be generated and directed into problem areas of cost, selling price, discount, profit and loss, etc.

C. Suggestions to teachers - It is suggested that the topic of consumer mathematics be considered at various times throughout the year when the opportunities arise. Consumer mathematics, however, should not be presented in this course in regular unit form. The topics of discussion within the subject of consumer mathematics are within the individual teacher's discretion, but the choice should reflect student

interest. Many helpful suggestions and ideas can be found in the Bureau of Secondary Curriculum Development's handbook on *Consumer Education, Materials for an Elective Course.*

UNIT XIV. Mathematical Reasoning

In this unit it is hoped that a system of reasoning will be developed based on the validity of an argument, remembering, however, that a false conclusion can be reached through valid arguments. Sets and Venn diagrams can be easily used in mathematical reasoning.

A. Simple sentence - A simple sentence is a declarative sentence that contains a subject and a predicate. It can be true, false, or undetermined.

I. Quantifiers - The first word of a sentence is often important in determining if a sentence is true or false.
   a. *All* rectangles are squares. False
   b. *Some* rectangles are squares. True
   c. *No* rectangle is a square. False

If the statement begins with	it is true if	it is false if
all - none,	every member fits the statement	one member does not fit the statement
some,	at least one member fits the statement	no member fits the statement

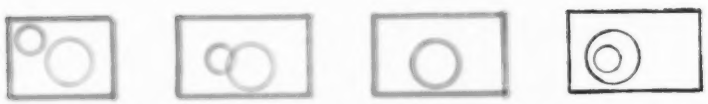
Example: A = {1,2,3} B = {1,2} C = {2,3} D = {2,4}
1. All members of D belong to A. False
2. No members of C belongs to B. False
3. Some members of C belong to D. True

B. Compound sentences - In mathematics, they are just like compound sentences in English. They are simple sentences joined together by using connectives.

1. Connectives
   a. Conjunction "and"
   b. Disjunction "or"
   c. Implication "if --- then ---"

2. Symbols
   a. ^ , &
   b. v -inclusive
   c. ~

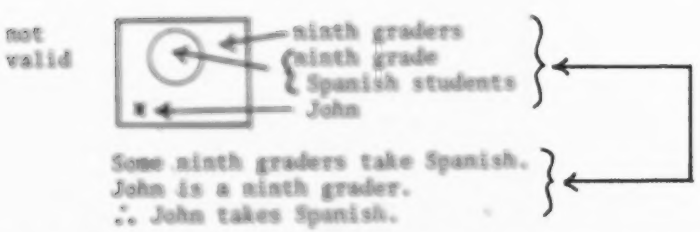
C. Venn diagrams - Review as needed. Two sets may be related in one of four ways.



disjoint overlapping equal subset

1. Examples
   a. All cows are mammals.
      All mammals are animals.
      ∴ All cows are animals.

   b.



D. Truth tables - Although Venn diagrams are useful to show simple arguments, truth tables are better for showing more complicated arguments.

1. Conjunction

P	Q	P∧Q
T	T	T
T	F	F
F	T	F
F	F	F

2. Disjunction

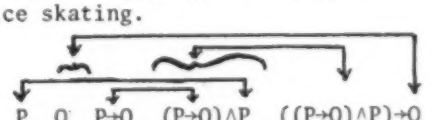
P	Q	P∨Q
T	T	T
T	F	T
F	T	T
F	F	F

3. Implication

P	Q	P→Q
T	T	T
T	F	F
F	T	T
F	F	T

4. Example: If the lake is frozen, we shall go ice skating. The lake is frozen. Therefore, we shall go ice skating.

P = the lake is frozen
Q = we shall go ice skating



P	Q	P→Q	(P→Q)∧P	((P→Q)∧P)→Q
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

E. Applications - The students should look for real life situations to apply mathematical reasoning in such areas as advertising, editorials, and their own arguments.

"Just because John eats Brand X, he is not necessarily a champion."

Teachers should be alert for examples of their students making statements indicating poor reasoning, and should capitalize on these to develop some of the preceding concepts.